

NON-ARCHIMEDEAN NORMAL FAMILIES

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ABSTRACT. We present several results on the compactness of the space of morphisms between analytic spaces in the sense of Berkovich. We show that under certain conditions on the source, every sequence of analytic maps having an affinoid target has a subsequence that converges pointwise to a continuous map. We also study the class of continuous maps that arise in this way. Locally, they turn analytic after a certain base change. We give some applications of these results to the dynamics of an endomorphism f of the projective space. We define the Fatou set as the normality locus of the family of the iterates $\{f^n\}$. We then generalize to the non-Archimedean setting a theorem of Ueda stating that every Fatou component is hyperbolically imbedded in the projective space.

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1. INTRODUCTION

The classical Montel's theorem asserts that any family of holomorphic functions on a domain in \mathbb{C}^n with values in a ball is relatively compact for the topology of the local uniform convergence [Mon07]. The proof uses Cauchy's estimates to obtain a uniform bound on the derivatives. By Ascoli-Arzelà's theorem the family then is equicontinuous and the result follows.

This result has several applications in complex dynamics. It also plays an important role in the study of Kobayashi hyperbolic complex analytic spaces. For instance, it is closely related to Zalcman's reparametrization lemma [Zal75], which is a key ingredient in the proof of Brody's Lemma [Bro78],

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characterizing compact Kobayashi hyperbolic complex analytic spaces in terms of the non-existence of entire curves.

The aim of this paper is to study the compactness properties of the space of morphisms between analytic spaces defined over a non-Archimedean complete field, in analogy to the classical Montel's Theorem. We therefore fix a non-Archimedean complete valued field k , which may be trivially valued.

An approach to this problem using equicontinuity has already been treated in the literature. Hsia gave in [Hsi00] an equicontinuity criterion for families of meromorphic functions on a disk. In [KS09], the Fatou set of a morphism of the projective space is defined as the equicontinuity locus of the family of iterates with respect to the chordal metric. However, this approach is limited by the fact that one cannot apply Ascoli-Arzelà's theorem in this context.

We will work on analytic spaces as defined in [Ber90, Ber93]. The main reason is that analytic spaces in the sense of Berkovich have good topological properties: they are locally compact and locally pathwise connected, what makes them a more adapted framework to arguments of analytic nature. The analytic spaces we shall be mostly interested in are Berkovich analytifications of projective varieties. Recall that the set of closed points of such a variety forms a dense subset of its analytification with empty interior. We shall refer to these points as *rigid points*. The previously mentioned equicontinuity results concern only the set of rigid points.

More recently, Favre, Kiwi and Trucco proved an analogue of Montel's theorem on the Berkovich analytic projective line $\mathbb{P}_k^{1,\text{an}}$, see [FKT12]. They show that when k is algebraically closed and has residue characteristic 0, then every sequence of analytic maps from *any* open connected subset X of $\mathbb{P}_k^{1,\text{an}}$ avoiding three points has a subsequence that is pointwise converging to a continuous map $X \rightarrow \mathbb{P}_k^{1,\text{an}}$. They made extensive use of Berkovich's geometry and their strategy benefits from the tree structure of $\mathbb{P}_k^{1,\text{an}}$.

We explore the higher dimensional case, and consequently use deeper facts from Berkovich theory. Of particular relevance for us is the theorem by Poineau stating that compact analytic spaces are sequentially compact, see [Poi13]. This result is non-trivial, since Berkovich spaces are not metrizable in general. We show:

Theorem A. *Let k be a non-Archimedean complete field and X a good, reduced, σ -compact, boundaryless strictly k -analytic space. Let Y be a k -affinoid space.*

Then, every sequence of analytic maps $f_n : X \rightarrow Y$ admits a pointwise converging subsequence whose limit is continuous.

The seemingly complicated hypothesis on the source space X are not such in fact. We refer the reader to §2 for a detailed discussion on the technical assumptions on X . For the moment, let us indicate that two important

classes of k -analytic spaces satisfy these properties: analytifications of algebraic varieties and connected components of the analytic interior of an affinoid space. The latter are the main examples of *basic tubes*. They have been thoroughly studied by Bosch and Poineau, see [Bos77, Poi14].

Remark that the boundaryless assumption is crucial, as problems arise even in the affinoid case. Consider for instance the sequence of analytic maps from the closed unit disk \mathbb{D} into itself $f_n : z \mapsto z^{n!}$. For every $n \in \mathbb{N}$, the Gauss point is a fixed point for f_n . One can show that f_n is pointwise converging, but its limit map is zero on the whole open unit disk and hence not continuous.

In view of Theorem A, we say that a family of analytic maps \mathcal{F} from a boundaryless k -analytic space X into a compact space Y is normal at a point $x \in X$ if for every sequence $\{f_n\}$ in \mathcal{F} there exists a neighbourhood $V \ni x$ and a subsequence f_{n_j} that is pointwise converging on V to some continuous map $f : V \rightarrow Y$.

We now turn to the problem of describing the limits of pointwise converging analytic maps. As opposed to the complex setting, one cannot expect the limit maps from Theorem A to be analytic. Indeed, when k is algebraically closed any constant map $f : X \rightarrow Y$, $f \equiv y \in Y$, can be realized as the limit of constant analytic maps. However, f is analytic if and only if y is rigid.

In spite of not being analytic in general, the continuous limit maps obtained in Theorem A are of a very particular kind: they turn analytic after a suitable base change. In order to specify this phenomenon precisely, we rely again in a crucial way on the results of Poineau. Let X be a k -analytic space. For every complete extension K of k , we denote by $\pi_{K/k} : X_K \rightarrow X$ the usual base change morphism. Every k -point in X defines a K -point in X_K in a natural manner. When the base field k is algebraically closed, Poineau [Poi13] shows that this inclusion admits a unique *continuous* extension $\sigma_{K/k} : X \rightarrow X_K$, which by construction defines a section of $\pi_{K/k}$.

Theorem B. *Let k be a non-Archimedean algebraically closed complete field and X a good, reduced, σ -compact, boundaryless strictly k -analytic space. Let Y be a k -affinoid space. Let $f_n : X \rightarrow Y$ be a sequence of analytic maps converging pointwise to a continuous map f .*

Then, for any point $x \in X$ one can find an affinoid neighbourhood Z of x , a complete extension K/k and a K -analytic map $F : Z_K \rightarrow Y_K$ such that

$$f|_Z = \pi_{K/k} \circ F \circ \sigma_{K/k}.$$

It would be interesting to find a K -analytic map F such that the stronger condition $\pi_{K/k} \circ F = f \circ \pi_{K/k}$ holds, but our proof does not show this.

Let us explain the proof of Theorem B in the case where X is the open r -dimensional polydisk \mathbb{D}_k^r and Y the closed s -dimensional polydisk $\bar{\mathbb{D}}_k^s$. The key idea is to view the set of all analytic maps from \mathbb{D}_k^r to $\bar{\mathbb{D}}_k^s$ as the set

of rigid points of an infinite dimensional polydisk $\text{Mor}_\infty^{r,s}$. This procedure can be easily illustrated in the polynomial case. Observe that a polynomial map sending \mathbb{D}^r into $\bar{\mathbb{D}}^s$ is given by finitely many coefficients in the base field k with norm at most 1, and so defines a rigid point in an appropriate dimensional closed unit polydisk. This procedure can be done similarly for general analytic maps. In this case, the coefficients define a rigid point in an infinite dimensional polydisk denoted $\text{Mor}_\infty^{r,s}$.

Now take a sequence $f_n : \mathbb{D}^r \rightarrow \bar{\mathbb{D}}^s$ of analytic maps, associated to a sequence of rigid points $\{\alpha^{(n)}\}$ in $\text{Mor}_\infty^{r,s}$. It can be showed that the fact that f_n converges pointwise to some continuous map f amounts for $\alpha^{(n)}$ to converging to some point α in $\text{Mor}_\infty^{r,s}$. Observe that α is not rigid in general, but after a base change by $\mathcal{H}(\alpha)$, the complete residue field at α , the point α can be lifted to a rigid point in $\text{Mor}_{\infty, \mathcal{H}(\alpha)}^{r,s}$. This point defines a $\mathcal{H}(\alpha)$ -analytic map $F : \mathbb{D}_{\mathcal{H}(\alpha)}^r \rightarrow \bar{\mathbb{D}}_{\mathcal{H}(\alpha)}^s$ that satisfies the equality $f = \pi_{\mathcal{H}(\alpha)/k} \circ F \circ \sigma_{\mathcal{H}(\alpha)/k}$. Observe that F is not canonical, as it depends on the choice of the rigid point in $\text{Mor}_{\infty, \mathcal{H}(\alpha)}^{r,s}$ lying over α .

We go beyond Theorem B and show that to any point α in $\text{Mor}_\infty^{r,s}$ one can associate a continuous map from \mathbb{D}^r to $\bar{\mathbb{D}}^s$ in a continuous way, in the sense that for any sequence of points $\alpha^{(n)}$ in $\text{Mor}_\infty^{r,s}$ converging to $\alpha \in \text{Mor}_\infty^{r,s}$, the corresponding sequence of continuous maps converges everywhere pointwise to the continuous map associated to α . In Section 4 we detail this correspondence.

This result suggests the following definition. We say that a continuous map f between analytic spaces is *weakly analytic* if it is locally of the form $f = \pi_{K/k} \circ F \circ \sigma_{K/k}$ for some complete extension K of k and some K -analytic map F . In fact, weakly analytic maps can be characterized as being locally the pointwise limit of analytic maps. In §5 we shall prove that weakly analytic maps share many properties with analytic maps, such as an isolated zero principle on curves.

We give applications of Theorem A to the dynamics of an endomorphism f of the k -analytic projective space $\mathbb{P}_k^{N, \text{an}}$ of degree at least 2. We define the Fatou set of such an endomorphism as the normality locus of the family of iterates $\{f^n\}_{n \in \mathbb{N}}$. Kawaguchi and Silverman associated a non-Archimedean Green function G_f to f in [KS07, KS09], generalizing the classical complex construction by Hubbard [Hub86] and Fornaess and Sibony [FS95]. We show in Theorem 6.10 that the Fatou set of f can be characterized in terms of the strong pluriharmonicity locus of the Green function G_f in the sense of Chambert-Loir [CL11].

There are two main results on the geometry of the Fatou set of an endomorphism of the complex projective space of degree at least 2, see [Sib99] for a complete survey. Every Fatou component is a Stein space [FS95] and is hyperbolically imbedded in $\mathbb{P}_{\mathbb{C}}^N$ in the sense of Kobayashi [Ued94].

Here we shall focus our attention on the hyperbolicity properties of the Fatou components in the non-Archimedean setting. To motivate our next result, recall that a subspace Ω of a complex analytic space Y is hyperbolically imbedded if the Kobayashi distance on Ω does not degenerate towards its boundary [Kob98, Lan87]. If Ω is relatively compact in Y , then Ω is hyperbolically imbedded in Y if and only if the family $\text{Hol}(\mathbb{D}, \Omega)$ of holomorphic maps from the open unit disc \mathbb{D} to Ω is relatively locally compact in $\text{Hol}(\mathbb{D}, Y)$, see [Lan87, §II, Theorem 1.2].

In our context, we prove:

Theorem C. *Let $f : \mathbb{P}^{N,\text{an}} \rightarrow \mathbb{P}^{N,\text{an}}$ be an endomorphism of degree at least 2. Let Ω be a Fatou component of f , and let U be any connected open subset of $\mathbb{P}^{1,\text{an}}$. Then, the family $\text{Mor}(U, \Omega)$ of analytic maps from U to Ω is normal.*

Note that in the non-Archimedean setting checking the normality for *every* open subset U of $\mathbb{P}^{1,\text{an}}$ is stronger than just for the open unit disk, as opposed to the complex case, see [Kob98, Theorem 5.1.5]. For instance, the family $\text{Mor}(\mathbb{D}, \mathbb{A}^{1,\text{an}} \setminus \{0\})$ is normal, whereas this is not true if one replaces the source by the punctured open unit disk.

It remains open whether in Theorem C one can take U to be any basic tube.

As a corollary of Theorem C we have the following Picard-type result:

Corollary D. *Let Ω be a Fatou component of an endomorphism $f : \mathbb{P}^{N,\text{an}} \rightarrow \mathbb{P}^{N,\text{an}}$ of degree $d \geq 2$. Then every analytic map from $\mathbb{A}^{1,\text{an}} \setminus \{0\}$ to Ω is constant.*

This paper is structured as follows. In Section 2 we review some basic facts about Berkovich spaces and summarize several results on universal points from [Poi13] that will be needed in the sequel. Section 3 comprises the proof of Theorem A. In Section 4 we describe the structure of the topological space that parametrizes the continuous maps that appear as pointwise limits of analytic maps between polydisks. We also give a proof of Theorem B. The properties of these maps are studied in §5. Finally, in Section §6 we give applications to dynamics of the previous results and prove Theorem C and Corollary D.

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2. GENERAL FACTS ON ANALYTIC SPACES

Throughout this paper, k is a field endowed with a non-Archimedean complete norm $|\cdot|$. We do not exclude the trivially valued case. Except in §3.5, k will be algebraically closed. We write $|k^\times| = \{|x| : x \in k^\times\} \subseteq \mathbb{R}_+$ for

its value group and $k^\circ = \{x \in k : |x| \leq 1\}$ for its ring of integers. The latter is a local ring with maximal ideal $k^{\circ\circ} = \{x \in k : |x| < 1\}$. The *residue field* of k is $\tilde{k} = k^\circ/k^{\circ\circ}$.

The basic reference for this section is Berkovich's original text [Ber90]. See also [Tem15] for a more recent survey.

2.1. Analytic spaces. Pick a positive integer N and an N -tuple of positive real numbers $r = (r_1, \dots, r_N)$. Denote by $k\{r^{-1}T\}$ the set of power series $f = \sum_I a_I T^I$, $I = (i_1, \dots, i_N)$, with coefficients $a_I \in k$ such that $|a_I| r^I \rightarrow 0$ as $|I| := i_1 + \dots + i_N$ tends to infinity. The norm $\|\sum_I a_I T^I\| = \max_I |a_I| r^I$ makes $k\{r^{-1}T\}$ into a Banach k -algebra. When $r = (1, \dots, 1)$, the previous algebra is called the Tate algebra and we denote it by \mathcal{T}_n .

Let $\varphi : \mathcal{B} \rightarrow \mathcal{A}$ be a morphism of Banach k -algebras. The residue norm on $\mathcal{B}/\text{Ker}\varphi$ is defined by $|a| = \inf_{\varphi(b)=a} |b|$, and we say that φ is *admissible* if the residue norm is equivalent to the restriction to the image of φ of the norm on \mathcal{A} .

A Banach k -algebra \mathcal{A} is called *affinoid* if there exists an admissible surjective morphism of k -algebras $k\{r^{-1}T\} \rightarrow \mathcal{A}$. If $r_i \in |k^\times|$ for all i , then \mathcal{A} is said to be *strictly affinoid*.

For any k -affinoid algebra \mathcal{A} , we denote by $X = \mathcal{M}(\mathcal{A})$ the set of all bounded multiplicative seminorms on \mathcal{A} whose restriction to k is the absolute value on k . Given $f \in \mathcal{A}$, its image under a seminorm $x \in \mathcal{M}(\mathcal{A})$ is denoted by $|f(x)| \in \mathbb{R}_+$. The set $\mathcal{M}(\mathcal{A})$ is called the *analytic spectrum* of \mathcal{A} and is endowed with the weakest topology such that all the functions of the form $x \mapsto |f(x)|$ with $f \in \mathcal{A}$ are continuous. The resulting topological space X is compact and naturally carries a sheaf of analytic functions \mathcal{O}_X such that $\mathcal{O}_X(X) = \mathcal{A}$, see [Ber90, §2.3]. The locally ringed space (X, \mathcal{O}_X) is called a k -affinoid space.

Given a point $x \in X = \mathcal{M}(\mathcal{A})$, the fraction field of $\mathcal{A}/\text{Ker}(x)$ naturally inherits from x a norm extending the one on k . Its completion is the *complete residue field* at x and denoted by $\mathcal{H}(x)$. When $\mathcal{H}(x)$ is a finite extension of k (or equivalently when $\mathcal{H}(x) = k$, since k is supposed to be algebraically closed), we say that x is *rigid*. The set $X(k)$ of rigid points of X is dense in X .

A character is a bounded homomorphism $\mathcal{A} \rightarrow K$, with K any complete extension of k . If L is a complete extension of K , we say that the characters $\mathcal{A} \rightarrow K \hookrightarrow L$ and $\mathcal{A} \rightarrow K$ are equivalent. Composing the character $\mathcal{A} \rightarrow K$ with the norm on K gives rise to a seminorm on \mathcal{A} that is bounded, and thus corresponds to a point $x \in \mathcal{M}(\mathcal{A})$. Equivalent characters give rise to the same point. Conversely, every point $x \in \mathcal{M}(\mathcal{A})$ induces a character $\chi_x : \mathcal{A} \rightarrow \mathcal{H}(x)$ in a natural way. Any other equivalent character $\mathcal{A} \rightarrow K$ giving rise to x can be decomposed as $\mathcal{A} \rightarrow \mathcal{H}(x) \hookrightarrow K$.

The closed polydisk of dimension N and polyradius $r = (r_1, \dots, r_N) \in (\mathbb{R}_*^+)^N$ is defined to be $\bar{\mathbb{D}}_k^N(r) := \mathcal{M}(k\{r^{-1}T\})$. When $r = (1, \dots, 1)$ we just write $\bar{\mathbb{D}}^N$, and when $N = 1$ we denote it by $\bar{\mathbb{D}}$. The Gauss point $x_g \in \bar{\mathbb{D}}$ is the point associated to the norm $|(\sum a_i T^i)(x_g)| := \max |a_i|$.

General analytic spaces are ringed spaces (X, \mathcal{O}_X) obtained by gluing together affinoid spaces. Difficulties arise in the gluing construction as affinoid spaces are compact, and we refer to [Ber90, Ber93] for a precise definition. Analytic spaces are locally compact and locally path-connected. Given an analytic space X , we denote by $|X|$ its underlying topological space.

The following topological result, due to Poineau, will be systematically used throughout the paper:

Theorem 2.1 ([Poi13]). *Every k -analytic space X is a Fréchet-Urysohn space. In particular, every compact subset of X is sequentially compact.*

In the following, we will always deal with *good* analytic spaces: these are locally ringed spaces modelled on affinoid spaces. In other words, any point in a good analytic space admits a neighbourhood isomorphic to an affinoid space.

For any point x in a k -analytic space X , the stalk $\mathcal{O}_{X,x}$ is a local k -algebra with maximal ideal \mathfrak{m}_x . It inherits an absolute value extending the one on k , and the completion of $\mathcal{O}_{X,x}/\mathfrak{m}_x$ is again called the *completed residue field of x* and denoted by $\mathcal{H}(x)$. In particular, when X is an affinoid space, this definition coincides with the previous one.

The open polydisk of dimension N and polyradius $r = (r_1, \dots, r_N) \in (\mathbb{R}_*^+)^N$ is the set

$$\mathbb{D}_k^N(r) = \{x \in \bar{\mathbb{D}}_k^N(r) : |T_i(x)| < r_i, i = 1, \dots, N\}.$$

It can be naturally endowed with a structure of good analytic space by writing it as the increasing union of N -dimensional polydisks $\mathbb{D}_k^N(\rho)$ whose radii $\rho = (\rho_1, \dots, \rho_N) \in (\mathbb{R}_*^+)^N$ satisfy $\rho_i < r_i$ for all $i = 1, \dots, N$.

2.2. Analytic maps. A morphism of k -affinoid spaces $\mathcal{M}(\mathcal{A}) \rightarrow \mathcal{M}(\mathcal{B})$ is by definition one induced by a bounded morphism of Banach k -algebras $\varphi : \mathcal{B} \rightarrow \mathcal{A}$. The fibre of $\mathcal{M}(\mathcal{A}) \rightarrow \mathcal{M}(\mathcal{B})$ over a point $y \in \mathcal{M}(\mathcal{B})$ is isomorphic to $\mathcal{M}(\mathcal{A} \hat{\otimes}_{\mathcal{B}} \mathcal{H}(y))$, see §2.6 for the notion of complete tensor product. Indeed, let $y \in \mathcal{M}(\mathcal{B})$ and let $\chi_y : \mathcal{B} \rightarrow \mathcal{H}(y)$ be the associated character. By definition, x is mapped to y if and only if the composite $\mathcal{B} \xrightarrow{\varphi} \mathcal{A} \rightarrow \mathcal{H}(x)$ factors through $\mathcal{H}(y)$. The latter is equivalent to the morphism χ_x factorizing through the \mathcal{B} -algebra morphism $\mathcal{A} \hat{\otimes}_{\mathcal{B}} \mathcal{H}(y) \rightarrow \mathcal{H}(x)$.

A morphism $\mathcal{M}(\mathcal{A}) \rightarrow \mathcal{M}(\mathcal{B})$ is a *closed immersion* when φ is surjective and admissible.

A surjective morphism $\varphi : \mathcal{T}_N \rightarrow \mathcal{A}$ is called *distinguished* if the quotient norm $|\cdot|_\varphi$ induced by φ agrees with the sup norm on \mathcal{A} , see [BGR84, §6.4.3].

We say that \mathcal{A} is distinguished if such an epimorphism exists. It can be shown that over an algebraically closed field k , every reduced algebra (i.e. without non-trivial nilpotents) is distinguished [BGR84, Theorem 6.4.3/1]. The key property of distinguished epimorphisms is that the reduction $\tilde{\mathcal{A}}$ is isomorphic to the quotient $\tilde{\mathcal{T}}_N / \ker(\varphi)$.

Given any two k -analytic spaces X and Y , we let $\text{Mor}(X, Y)$ be the set of all analytic maps from X to Y .

2.3. Analytification of algebraic varieties. To every algebraic variety X over k one can associate a k -analytic space X^{an} in a functorial way; see [Ber90, §3.4] for a detailed construction.

In the case of an affine variety $X = \text{Spec}(A)$, where A is a finitely generated k -algebra, then the set X^{an} consists of all the multiplicative seminorms on A whose restriction to k coincides with the norm on k . This set is endowed with the weakest topology such that all the maps of the form $x \in X^{\text{an}} \mapsto |f(x)|$ with $f \in A$ are continuous. Observe that any k -point $x \in X$ corresponds to a morphism of k -algebras $A \rightarrow k$ and its composition with the norm on k defines a rigid point in X^{an} . Since k is algebraically closed, one obtains in this way an identification of the set of closed points in X with the set of rigid points in X^{an} .

The analytification of a general algebraic variety X given by an affine open cover is obtained by glueing together the analytification of its affine charts in natural way. Analytifications of algebraic varieties are good analytic spaces, and closed points are in natural bijection with rigid points as in the affine case.

2.4. Boundary and interior. Any k -analytic space X comes with natural notions of boundary and interior, which are defined as follows.

A point x in an affinoid space X lies in the *interior* of X if there exists a closed immersion $\varphi : X \rightarrow \mathbb{D}^N(r)$ for some polyradius r and some integer N such that $\varphi(x)$ lies in the open polydisk $\mathbb{D}^N(r)$.

If X is an analytic space, a point x belongs to its interior if it admits an affinoid neighbourhood U such that x belongs to the interior of U . We let $\text{Int}(X)$ be the open set consisting of all the interior points in X . Its complement $\partial(X)$ is called the boundary of X . It is a closed subset.

The analytification of an algebraic variety is boundaryless.

In the remaining of this section, we explain how to compute the interior of a strictly k -affinoid space $X = \mathcal{M}(\mathcal{A})$. Recall that the spectral radius of $f \in \mathcal{A}$ is defined by

$$\rho(f) = \lim_{n \rightarrow \infty} \|f^n\|^{1/n},$$

where $\|\cdot\|$ is the Banach norm on \mathcal{A} . When \mathcal{A} is reduced, then ρ is a norm equivalent to $\|\cdot\|$. It follows that $\mathcal{A}^\circ = \{f \in \mathcal{A} : \rho(f) \leq 1\}$ is a local ring whose maximal ideal is equal to $\mathcal{A}^{\circ\circ} = \{f \in \mathcal{A} : \rho(f) < 1\}$. The reduction of \mathcal{A} is then defined as $\tilde{\mathcal{A}} := \mathcal{A}^\circ / \mathcal{A}^{\circ\circ}$, and the reduction of X is $\tilde{X} = \text{Spec}(\tilde{\mathcal{A}})$.

Observe that Noether's normalization Lemma [BGR84, Corollary 6.1.2/2] implies that for any strictly k -affinoid algebra \mathcal{A} , the reduction $\tilde{\mathcal{A}}$ is a finitely generated \tilde{k} -algebra, and thus \tilde{X} is an affine variety over the residue field \tilde{k} . The reduction of the closed polydisk $\bar{\mathbb{D}}_k^N$ is the affine space $\mathbb{A}_{\tilde{k}}^N$.

The reduction map $\text{red} : X \rightarrow \tilde{X}$ is defined as follows. Every bounded morphism of Banach k -algebras $\mathcal{A} \rightarrow \mathcal{B}$ induces a morphism between their reductions $\tilde{\mathcal{A}} \rightarrow \tilde{\mathcal{B}}$. In particular, from the character $\chi_x : \mathcal{A} \rightarrow \mathcal{H}(x)$ associated to a point $x \in X$ we obtain a \tilde{k} -algebra morphism $\tilde{\chi}_x : \tilde{\mathcal{A}} \rightarrow \widetilde{\mathcal{H}(x)}$. We set $\text{red}(x) := \text{Ker}(\tilde{\chi}_x)$. This map is anticontinuous.

Lemma 2.2. *Let X be a strictly k -affinoid space. Then,*

$$\text{Int}(X) = \{x \in X : \text{red}(x) \text{ is a closed point}\}.$$

Proof. Let $\varphi : X \rightarrow \bar{\mathbb{D}}^N$ be a closed immersion. We have the following diagram:

$$\begin{array}{ccc} X & \xrightarrow{\varphi} & \bar{\mathbb{D}}^N \\ \text{red} \downarrow & & \downarrow \text{red} \\ \tilde{X} & \xrightarrow{\tilde{\varphi}} & \mathbb{A}_{\tilde{k}}^N \end{array}$$

Let $x \in X$. If its reduction $\tilde{x} = \text{red}(x)$ is a closed point then so is $\tilde{\varphi}(\tilde{x})$. The inverse image of $\tilde{\varphi}(\tilde{x})$ is isomorphic to an open polydisk \mathbb{D}^N , and the commutativity of the diagram implies that $\varphi(x)$ lies in \mathbb{D}^N .

Let \mathcal{A} be the underlying affinoid algebra of X and pick a point $x \in \text{Int}(X)$. By [Ber90, Proposition 2.5.2], the image of the morphism of \tilde{k} -algebras $\tilde{\chi}_x : \tilde{\mathcal{A}} \rightarrow \widetilde{\mathcal{H}(x)}$ induced by χ_x is integral over \tilde{k} . This implies that $\tilde{\chi}_x(\tilde{\mathcal{A}}) \simeq \tilde{\mathcal{A}}/\text{Ker}(\tilde{\chi}_x)$ is a field. Thus, \tilde{x} is a closed point of \tilde{A} . \square

Proposition 2.3. *Let $X = \mathcal{M}(\mathcal{A})$ and $Y = \mathcal{M}(\mathcal{B})$ be strictly k -affinoid spaces, and let $f : X \rightarrow Y$ be a finite morphism. Then, $\text{Int}(X) = f^{-1}(\text{Int}(Y))$.*

Proof. The morphism $f : X \rightarrow Y$ induces the following commutative diagram:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \text{red} \downarrow & & \downarrow \text{red} \\ \text{Spec}(\tilde{\mathcal{A}}) & \xrightarrow{\tilde{f}} & \text{Spec}(\tilde{\mathcal{B}}) \end{array}$$

Let x be a point in $\text{Int}(X)$. By Lemma 2.2, its image $f(x)$ belongs to $\text{Int}(Y)$.

Let now $x \in X$ be such that $f(x) = y$ lies in $\text{Int}(Y)$. By the previous lemma, we have to show that $\text{red}(x)$ is a closed point of \tilde{X} . Consider the ring homomorphism $\varphi : \tilde{\mathcal{B}} \rightarrow \tilde{\mathcal{A}}$ inducing \tilde{f} . It induces a morphism $\varphi' : \tilde{B}/\text{ker}(\tilde{\chi}_y) \rightarrow \tilde{A}/\text{ker}(\tilde{\chi}_x)$, as the diagram above is commutative. Observe that φ is integral, since it is finite ([BGR84, Theorem 6.3.5/1]), and thus φ' is also integral. As $y \in \text{Int}(Y)$, by Lemma 2.2 the quotient $\tilde{B}/\text{ker}(\tilde{\chi}_y)$ is a

field. This implies that $\tilde{\mathcal{A}}/\ker(\chi_x)$ is a field and thus that $\text{red}(x)$ is a closed point. \square

Remark 2.4. *The previous results hold for any (not necessarily strictly) affinoid space. In that case, one needs to use Temkin's graded reduction of affinoid algebras ([Tem00, Tem04]).*

2.5. Basic tubes. We introduce the following terminology.

Definition 2.5. *A k -analytic space X is called a basic tube if there exists a reduced equidimensional strictly k -affinoid space \hat{X} and a closed point \tilde{x} in its reduction such that X is isomorphic to $\text{red}^{-1}(\tilde{x})$.*

By convention, a basic tube is therefore reduced.

Example 2.6. *Let a_1, \dots, a_m be non-rigid points in $\mathbb{P}^{1, \text{an}}$. Then every connected component of $\mathbb{P}^{1, \text{an}} \setminus \{a_1, \dots, a_m\}$ is a basic tube.*

Recall that a topological space is σ -compact if it is the union of countably many compact subspaces. For instance, open Berkovich polydisks or the analytification of an algebraic variety are σ -compact spaces. However, note that σ -compactness is a non-trivial assumption, since examples of non σ -compact spaces include the closed disk with the Gauss point removed for a base field k with uncountable reduction \tilde{k} .

Theorem 2.7. *A basic tube is connected, has no boundary and is σ -compact.*

The fact that any basic tube is connected is a deep theorem due to [Bos77] and [Poi14]. The other two statements follow from the next proposition of independent interest.

Proposition 2.8. *For every basic tube X there exist a strictly k -affinoid space \hat{X} and a distinguished closed immersion into some closed polydisk $\hat{X} \rightarrow \mathbb{D}^N$ such that X is isomorphic to $\hat{X} \cap \mathbb{D}^N$.*

Proof. Let $\hat{X} = \mathcal{M}(\mathcal{A})$ be an equidimensional reduced k -affinoid space and let \tilde{x} be a closed point in its reduction such that $\text{red}^{-1}(\tilde{x})$ is isomorphic to X . Recall from §2.2 that there exists a distinguished closed immersion $\varphi : \hat{X} \rightarrow \mathbb{D}^N$, as k is algebraically closed and \mathcal{A} is reduced.

Hence, $\tilde{\mathcal{A}}$ is isomorphic to $\tilde{k}[T_1, \dots, T_N]/\widetilde{\ker(\varphi)}$ by [BGR84, Proposition 6.4.3/4], and the induced morphism $\text{Spec}(\tilde{\mathcal{A}}) \rightarrow \mathbb{A}_{\tilde{k}}^N$ is a closed immersion by [BGR84, Theorem 6.3.1/6]. We may assume that \tilde{x} is mapped to 0. We conclude that x is mapped to a point in $\text{red}^{-1}(0)$, which is isomorphic to \mathbb{D}^N by [BL85, Proposition 2.2]. \square

2.6. Universal points and base changes. Let \mathcal{A} and \mathcal{B} be two Banach k -algebras and denote by $|\cdot|_{\mathcal{A}}$ and $|\cdot|_{\mathcal{B}}$ their respective Banach norms. On the tensor product $\mathcal{A} \otimes_k \mathcal{B}$ we have the seminorm that associates to every $f \in \mathcal{A} \otimes_k \mathcal{B}$ the quantity

$$\|f\| = \inf \max |a_i|_{\mathcal{A}} \cdot |b_i|_{\mathcal{B}},$$

where the infimum is taken over all the possible expressions of f of the form $f = \sum_i a_i \otimes b_i$ with $a_i \in \mathcal{A}$ and $b_i \in \mathcal{B}$. The seminorm $\|\cdot\|$ induces the *tensor norm* on quotient the $\mathcal{A} \otimes_k \mathcal{B} / \{\|f\| = 0\}$, whose completion is a Banach k -algebra satisfying a suitable natural universal property. This algebra is called the *complete tensor product* of \mathcal{A} and \mathcal{B} and we denote it by $\mathcal{A} \hat{\otimes}_k \mathcal{B}$, see [BGR84, §2.1.7].

Given a k -affinoid algebra \mathcal{A} and a complete extension K of k , the K -algebra $\mathcal{A} \hat{\otimes}_k K$ is in fact K -affinoid, thus we may define the scalar extension of the k -affinoid space $X = \mathcal{M}(\mathcal{A})$ by K as the K -affinoid space $X_K := \mathcal{M}(\mathcal{A} \hat{\otimes}_k K)$. The natural morphism $\mathcal{A} \rightarrow \mathcal{A} \hat{\otimes}_k K$ induces a base change morphism $\pi_{K/k} : X_K \rightarrow X$. This construction can be done similarly for general k -analytic spaces.

Recall the following definition from [Ber90, Poi13]:

Definition 2.9. *Let X be a k -analytic space. A point x in X is universal if for every complete extension K of k the tensor norm on $\mathcal{H}(x) \hat{\otimes}_k K$ is multiplicative.*

The key feature of universal points is that they can be canonically lifted to any scalar extension. To explain this fact we may suppose that X is an affinoid space with underlying algebra \mathcal{A} . Pick any universal point $x \in X$ and fix any complete extension K of k . The k -algebra morphism $\mathcal{A} \rightarrow \mathcal{H}(x)$ corresponding to $x \in X$ induces a K -algebra morphism $\mathcal{A} \hat{\otimes}_k K \rightarrow \mathcal{H}(x) \hat{\otimes}_k K$.

Since x is universal, the tensor norm on $\mathcal{H}(x) \hat{\otimes}_k K$ is multiplicative and so the composite of $\mathcal{A} \hat{\otimes}_k K \rightarrow \mathcal{H}(x) \hat{\otimes}_k K$ with the tensor norm defines a point in X_K . The point in X_K obtained by these means is denoted by $\sigma_{K/k}(x)$.

Observe that if $x \in X$ is rigid, then so is $\sigma_{K/k}(x)$, and that $\sigma_{K/k}$ is a section of $\pi_{K/k}$ on the set of universal points of X .

Theorem 2.10 ([Poi13]). *Let k be an algebraically closed complete field and X a k -analytic space. Then, every point $x \in X$ is universal, and the map $\sigma_{K/k} : X \rightarrow X_K$ is continuous.*

We conclude this section by recalling the following construction.

Lemma 2.11. *Let X be a good k -analytic space and x a point in X . Then for every complete extension K of $\mathcal{H}(x)$, the fibre $\pi_{K/k}^{-1}(x)$ contains a rigid point.*

Proof. Pick a point $x \in X$. We may suppose $K = \mathcal{H}(x)$. Since the statement is local at x , we may replace X by any neighbourhood of x . The k -analytic space X being good, we may suppose it is an affinoid space. Denote by \mathcal{A} the underlying k -affinoid algebra. Consider the character $\chi_x : \mathcal{A} \rightarrow \mathcal{H}(x)$. The morphism $\mathcal{A} \hat{\otimes}_k \mathcal{H}(x) \rightarrow \mathcal{H}(x)$ sending $f \otimes a$ to $\chi_x(f) \cdot a$ is by definition a rigid point in $X_{\mathcal{H}(x)}$ lying over x . \square

We shall denote by $\tau(x) \in X_{\mathcal{H}(x)}$ the rigid point lying over $x \in X$ obtained in the previous proof. This point $\tau(x)$ is not to be confused with $\sigma_{K/k}(x)$.

3. MONTEL-TYPE RESULTS

The aim of this section is to prove Theorem A. To that end, we fix a non-Archimedean complete valued field k that is algebraically closed. Under suitable assumptions on the k -analytic spaces X and Y we shall prove that any sequence of analytic maps $f_n : X \rightarrow Y$ admits a subsequence with continuous limit.

We shall do this in several steps: first we show how to extract subsequences that converge pointwise; then we prove the continuity of the limit.

3.1. Pointwise convergence on open polydisks. In this section, we prove:

Theorem 3.1. *Let X be a basic tube defined over k and let Y be a k -affinoid space.*

For any sequence of analytic maps $f_n : X \rightarrow Y$, there exists a subsequence that converges pointwise everywhere on X .

Given a multi-index $I = (i_1, \dots, i_r)$, denote by $|I| = \max_j i_j$.

Proof. Let us first treat the case $X = \mathbb{D}^r$. Consider a sequence of analytic maps $f_n : \mathbb{D}^r \rightarrow Y$. Since any k -affinoid space can be embedded into a polydisk, we may readily assume that $Y = \bar{\mathbb{D}}^s$ for some integer s .

Every analytic map $f_n : \mathbb{D}^r \rightarrow \bar{\mathbb{D}}^s$ is of the form $f_n = (f_1^{(n)}, \dots, f_s^{(n)})$, with $f_l^{(n)} = \sum_I a_{l,I}^{(n)} T^I$, where $I = (i_1, \dots, i_r)$, $|a_{l,I}^{(n)}| \leq 1$ and $|a_{l,I}^{(n)}| \rho^I \xrightarrow{|I| \rightarrow \infty} 0$ for every $0 < \rho < 1$. For every $l = 1, \dots, s$, we set $\|f_l^{(n)}\| = \max_I |a_{l,I}^{(n)}| \leq 1$.

For every integer δ , we introduce the truncated maps

$$f_n^\delta = (f_{1,\delta}^{(n)}, \dots, f_{s,\delta}^{(n)}) = \left(\sum_{|I| \leq \delta} a_{1,I}^{(n)} T^I, \dots, \sum_{|I| \leq \delta} a_{s,I}^{(n)} T^I \right).$$

Observe that since $|a_{l,I}^{(n)}| \leq 1$, the points

$$\alpha_{n,\delta} := \left((a_{1,I}^{(n)})_{|I| \leq \delta}, \dots, (a_{s,I}^{(n)})_{|I| \leq \delta} \right) \quad (3.1)$$

are rigid points and belong to $\bar{\mathbb{D}}^{(\delta+1)rs}(k)$ for all n .

The polydisk $\bar{\mathbb{D}}_k^{(\delta+1)rs}$ is sequentially compact by Theorem 2.1, therefore the sequence $\{\alpha_{n,\delta}\}_n$ has a converging subsequence for every $\delta \geq 0$.

By a diagonal extraction argument, and possibly by replacing f_n by a subsequence, we may thus suppose that for every δ there exists $\alpha_\delta \in \bar{\mathbb{D}}_k^{(\delta+1)rs}$ such that $\alpha_{n,\delta} \rightarrow \alpha_\delta$.

Pick any $z \in \mathbb{D}^r$. Our goal is to show that the sequence $f_n(z)$ converges. Observe that this is equivalent to checking that for every $g \in \mathcal{T}_s$, the sequence of real numbers $\{|g(f_n(z))|\}_{n \in \mathbb{N}}$ is converging.

If z is a non-rigid point in \mathbb{D}^r , we make a base change by $\mathcal{H}(z)$ and take a rigid point $x \in \mathbb{D}_{\mathcal{H}(z)}^r$ lying over z (see Lemma 2.11). The maps f_n induce analytic maps $X_{\mathcal{H}(z)} \rightarrow Y_{\mathcal{H}(z)}$ and g defines an analytic function on $Y_{\mathcal{H}(z)}$. By definition,

$$|g(f_n(z))| = |g(f_n(\pi_{\mathcal{H}(z)/k}(x)))| = |g(f_n(x))|,$$

so that $|g(f_n(z))|$ converges if and only if $|g(f_n(x))|$ converges.

We may thus suppose that z is a rigid point. We fix a positive real number $\epsilon > 0$, and expand $g = \sum_J g_J T^J$ in the Tate algebra \mathcal{T}_s . We also consider a sufficiently large integer $d \geq 0$ such that $\max_{|J| \geq d+1} |g_J| < \epsilon$. Then the truncation $\tilde{g} = \sum_{|J| \leq d} g_J T^J$ satisfies

$$\begin{aligned} ||g(f_n(z))| - |\tilde{g}(f_n(z))|| &\leq |g(f_n(z)) - \tilde{g}(f_n(z))| \\ &= \left| \sum_{|J| \geq d+1} g_J \prod_{l=1}^s (f_l^{(n)}(z))^{j_l} \right| \\ &\leq \max_{|J| \geq d+1} |g_J| \prod_{l=1}^s \|f_l^{(n)}\|^{j_l} \leq \max_{|J| \geq d+1} |g_J| < \epsilon \end{aligned}$$

It follows that if $|\tilde{g}(f_n(z))|$ is a Cauchy sequence for all truncations of arbitrarily large degree d , then $|g(f_n(z))|$ will also be a Cauchy sequence.

In other words, if $|g(f_n(z))|$ converges when g is a polynomial, then it also does for any $g \in \mathcal{T}_s$. From now we may and shall assume that g is a polynomial of degree d .

Recall the definition of the truncated maps $f_n^\delta = (f_{1,\delta}^{(n)}, \dots, f_{s,\delta}^{(n)})$. We have:

$$|g(f_n^\delta(z))| = \left| \sum_{|J| \leq d} g_J \prod_{l=1}^s (f_{l,\delta}^{(n)}(z))^{j_l} \right| = \left| \sum_{|J| \leq d} g_J \prod_{l=1}^s \left(\sum_{|I| \leq \delta} a_{l,I}^{(n)} z^I \right)^{j_l} \right| = (*)$$

Taking the polynomial in $(\delta + 1)rs$ -variables

$$R := \sum_{|J| \leq d} g_J \prod_{l=1}^s \left(\sum_{|I| \leq \delta} S_{l,I} z^I \right)^{j_l} \in k[\{S_{l,I}\}_{1 \leq l \leq s, |I| \leq \delta}], \quad (3.2)$$

one sees that $(*) = |R(\alpha_{n,\delta})|$, and so $|R(\alpha_{n,\delta})| \rightarrow |R(\alpha_\delta)|$ as n tends to infinity since $\alpha_{n,\delta} \rightarrow \alpha_\delta$. In particular, the sequence $|g(f_n^\delta(z))|$ converges.

Pick any $\rho < 1$ such that z is a rigid point in $\mathbb{D}^r(\rho)$, and denote by $\|\cdot\|$ the Banach norm on $k\{\rho^{-1}T_1, \dots, \rho^{-1}T_r\}$. Observe that

$$\|f_l^{(n)} - f_{l,\delta}^{(n)}\| = \left\| \sum_{|I| \geq \delta+1} a_{l,I}^{(n)} T^I \right\| = \max_{|I| \geq \delta+1} |a_{l,I}^{(n)}| \rho^{|I|} \leq \rho^{\delta+1}.$$

Since g is a polynomial, we may assume its coefficients have norm at most 1. To simplify notation, set $f_{s+1}^{(n)} \equiv 1$, $f_{0,\delta}^{(n)} \equiv 1$ for all δ and for all n . We have:

$$\begin{aligned}
|g(f_n(z)) - g(f_n^\delta(z))| &= \left| \sum_J g_J \left[(f_n(z))^J - (f_n^\delta(z))^J \right] \right| \\
&= \left| \sum_J g_J \left[\sum_{l=1}^s \left((f_l^{(n)}(z))^{j_l} - (f_{l,\delta}^{(n)}(z))^{j_l} \right) \prod_{i=l+1}^s (f_i^{(n)}(z))^{j_i} \prod_{k=0}^{l-1} (f_{k,\delta}^{(n)}(z))^{j_k} \right] \right| \\
&\leq \max_{J,l} |g_J| \cdot \left| (f_l^{(n)}(z))^{j_l} - (f_{l,\delta}^{(n)}(z))^{j_l} \right| \prod_{i=l+1}^s |f_i^{(n)}(z)|^{j_i} \prod_{k=0}^{l-1} |f_{k,\delta}^{(n)}(z)|^{j_k} \\
&\leq \max_{J,l} |g_J| \cdot \left| (f_l^{(n)}(z))^{j_l} - (f_{l,\delta}^{(n)}(z))^{j_l} \right| \leq \max_{J,l} |g_J| \cdot |f_l^{(n)}(z) - f_{l,\delta}^{(n)}(z)| \\
&= \max_{J,l} |g_J| \cdot \left| \sum_{|I| \geq \delta+1} a_{l,I}^{(n)} z^I \right| \leq \max_J |g_J| \prod_{l=1}^s \max_{|I| \geq \delta+1} |a_{l,I}^{(n)}| \rho^{|I|},
\end{aligned}$$

so that

$$|g(f_n(z)) - g(f_n^\delta(z))| \leq \rho^{\delta+1}. \quad (3.3)$$

Write

$$\begin{aligned}
||g(f_n(z))| - |g(f_m(z))|| &\leq \max \left\{ ||g(f_n(z))| - |g(f_n^\delta(z))||, \right. \\
&\quad \left. ||g(f_n^\delta(z))| - |g(f_m^\delta(z))||, ||g(f_m^\delta(z))| - |g(f_m(z))|| \right\}.
\end{aligned}$$

The first and third terms are $\leq \rho^{\delta+1}$ by (3.3) and the second one tends to 0 by our preceding argument. If $\epsilon > 0$ is fixed, then we may take δ large enough such that $\rho^{\delta+1} \leq \epsilon$, and for any n, m large enough we get $||g(f_n(z))| - |g(f_m(z))|| \leq \epsilon$. It follows that $\{|g(f_n(z))|\}$ is a Cauchy sequence, concluding the proof of the theorem in the case $X = \mathbb{D}^r$.

Suppose now that X is a basic tube. Let \hat{X} be a k -affinoid space and $\hat{X} \rightarrow \bar{\mathbb{D}}^r$ a distinguished closed immersion such that X is isomorphic to $\hat{X} \cap \mathbb{D}^r$ (cf. Proposition 2.8). We may thus write X as a growing countable union of affinoid spaces $X = \bigcup_{0 < \rho < 1} X_\rho$. As the affinoid algebra corresponding to \hat{X} is isomorphic to the quotient of the Tate algebra \mathcal{T}_r by some closed ideal I , we may assume that the affinoid algebra \mathcal{A}_ρ of each X_ρ is of the form $k\{\rho^{-1}T_1, \dots, \rho^{-1}T_r\}$ modulo the ideal I . In particular, we have distinguished closed immersions $X_\rho \rightarrow \bar{\mathbb{D}}^r(\rho)$.

Let $f_n : X \rightarrow \bar{\mathbb{D}}^s$ be a sequence of analytic maps. For every $\rho < 1$, the restriction of each f_n to X_ρ can be extended to an analytic map $\bar{\mathbb{D}}^r(\rho) \rightarrow \bar{\mathbb{D}}^s$. Indeed, $f_n|_{X_\rho}$ is given by $f_1^{(n)}, \dots, f_s^{(n)} \in \mathcal{A}_\rho$. As for every $\rho < 1$ we have a distinguished epimorphism $k\{\rho^{-1}T_1, \dots, \rho^{-1}T_r\} \rightarrow \mathcal{A}_\rho$, we may lift each $f_l^{(n)}$, $l = 1, \dots, s$ to an element in $k\{\rho^{-1}T_1, \dots, \rho^{-1}T_r\}$ having the same norm. We conclude by the previous case and a diagonal extraction argument. \square

3.2. Fields with countable residue field. We observe in this section that Theorem 3.1 extends to maps between any k -affinoid spaces when the residue field of k is countable. This section will not be used in the rest of the paper, since the limits we obtain this way are not necessarily continuous.

Recall that the boundary of an affinoid can be written as a finite union of affinoid spaces defined over some extension of k , see [Duc12, Lemma 3.1]. Here we shall only use the following observation. Consider the closed N -dimensional polydisk $\bar{\mathbb{D}}^N$, and denote by $p_i : \bar{\mathbb{D}}^N \rightarrow \bar{\mathbb{D}}$ the projection to the i -th coordinate. Recall that the boundary of $\bar{\mathbb{D}}$ consists only of the Gauss point. It follows from Lemma 2.2 that the interior of $\bar{\mathbb{D}}^N$ is contained in $p_i^{-1}(\text{Int}(\bar{\mathbb{D}}))$ for every $i = 1, \dots, N$. Let now x be a point in $\partial\bar{\mathbb{D}}^N$ and consider the commutative diagram:

$$\begin{array}{ccc} \bar{\mathbb{D}}^N & \xrightarrow{p_i} & \bar{\mathbb{D}} \\ \text{red} \downarrow & & \downarrow \text{red} \\ \mathbb{A}_{\tilde{k}}^N & \xrightarrow{\tilde{p}_i} & \mathbb{A}_{\tilde{k}}^1 \end{array}$$

Suppose that $p_i(x) \neq x_g$ for all i . The diagram implies that there exist $\zeta_1, \dots, \zeta_N \in \tilde{k}$ such that every ideal of $\tilde{k}[T_1, \dots, T_N]$ of the form $\langle T_i - \zeta_i \rangle$ is contained in the prime ideal corresponding to $\text{red}(x)$. As a consequence, $\text{red}(x) \in \mathbb{A}_{\tilde{k}}^N$ is closed, contradicting the fact that x belongs to $\partial\bar{\mathbb{D}}^N$.

The boundary of $\bar{\mathbb{D}}^N$ is thus equal to the union $p_1^{-1}(x_g) \cup \dots \cup p_N^{-1}(x_g)$. Observe that each fibre $p_i^{-1}(x_g)$ is isomorphic to $\bar{\mathbb{D}}_{\mathcal{H}(x_g)}^{N-1}$.

Proposition 3.2. *Suppose k is a non-Archimedean complete valued field that is algebraically closed and such that \tilde{k} is countable. Let X and Y be k -affinoid spaces. Then, every sequence of analytic maps $f_n : X \rightarrow Y$ has an everywhere pointwise converging subsequence.*

Proof. We may assume $X = \bar{\mathbb{D}}_k^r$, $Y = \bar{\mathbb{D}}_k^s$ as in the proof of Theorem 3.1. The set of connected components of the interior of $\bar{\mathbb{D}}_k^r$ is in bijection with the set of \tilde{k} -points on its reduction $\mathbb{A}_{\tilde{k}}^r$ and hence is countable.

We now argue inductively on r . When $r = 1$, then the boundary of $\bar{\mathbb{D}}$ consists of a single point, namely the Gauss point. We may therefore apply Theorem 3.1 to each of the (countably many) components of the interior of $\bar{\mathbb{D}}$ and apply a diagonal extraction argument to conclude.

Assume now that the statement holds for the polydisk of dimension $r - 1$ defined over *any* complete valued field with countable residue field, and pick a sequence of analytic maps $f_n : \bar{\mathbb{D}}_k^r \rightarrow \bar{\mathbb{D}}_k^s$. As before, we apply Theorem 3.1 to each of the (countably many) components of the interior of $\bar{\mathbb{D}}_k^r$ so that we may suppose that f_n converges pointwise on the interior of $\bar{\mathbb{D}}_k^r$.

The boundary of $\bar{\mathbb{D}}_k^r$ is the union of r unit polydisks of dimension $r - 1$ defined over the field $\mathcal{H}(x_g)$ by our previous discussion. On each of these

we may apply the induction hypothesis, as the field $\widetilde{\mathcal{H}(x_g)}$ is isomorphic to $\tilde{k}(T)$, which is countable. This concludes the proof. \square

3.3. Polynomial maps of bounded degree. Pick any integers $r, s > 0$, $\delta \geq 0$ and fix a point α in the (Berkovich) analytic space $\bar{\mathbb{D}}_k^{(\delta+1)rs}$. We shall associate to these data a continuous map

$$P_\alpha = P_\alpha^{r,s} : \mathbb{A}_k^{r,\text{an}} \rightarrow \mathbb{A}_k^{s,\text{an}}.$$

Consider first the analytic map $\Phi : \bar{\mathbb{D}}_k^{(\delta+1)rs} \times \mathbb{A}_k^{r,\text{an}} \rightarrow \mathbb{A}_k^{s,\text{an}}$, given by the k -algebra morphism

$$\begin{aligned} k[T_1, \dots, T_s] &\rightarrow k[T_1, \dots, T_r] \{ (a_{1,I})_{|I| \leq \delta}, \dots, (a_{s,I})_{|I| \leq \delta} \} \\ T_l &\mapsto \sum_{|I| \leq \delta} a_{l,I} T^I. \end{aligned}$$

Next, consider the projection $\pi_1 : \bar{\mathbb{D}}_k^{(\delta+1)rs} \times \mathbb{A}_k^{r,\text{an}} \rightarrow \bar{\mathbb{D}}_k^{(\delta+1)rs}$. The fibre over the point $\alpha \in \bar{\mathbb{D}}_k^{(\delta+1)rs}$ is isomorphic to $\mathbb{A}_{\mathcal{H}(\alpha)}^{r,\text{an}}$ (cf. §2.2). Recall that the point $\alpha \in \bar{\mathbb{D}}_k^{(\delta+1)rs}$ is associated to the character $\chi_\alpha : k \{ (a_{1,I})_{|I| \leq \delta}, \dots, (a_{s,I})_{|I| \leq \delta} \} \rightarrow \mathcal{H}(\alpha)$. Set $K := \mathcal{H}(\alpha)$. The inclusion $\iota_K : \mathbb{A}_K^{r,\text{an}} \rightarrow \bar{\mathbb{D}}_k^{(\delta+1)rs} \times \mathbb{A}_k^{r,\text{an}}$ is given by

$$\begin{aligned} k[T_1, \dots, T_r] \{ (a_{1,I})_{|I| \leq \delta}, \dots, (a_{s,I})_{|I| \leq \delta} \} &\rightarrow K[T_1, \dots, T_r] \\ T_i &\mapsto T_i \\ a_{l,I} &\mapsto \chi_\alpha(a_{l,I}). \end{aligned}$$

Finally, for every $z \in \mathbb{A}_k^{r,\text{an}}$ we set:

$$P_\alpha(z) = \Phi \circ \iota_K \circ \sigma_{K/k}(z),$$

where $\sigma_{K/k} : \mathbb{A}_k^{r,\text{an}} \rightarrow \mathbb{A}_K^{r,\text{an}}$ is the canonical lift discussed in §2.6. The map $P_\alpha : \mathbb{A}_k^{r,\text{an}} \rightarrow \mathbb{A}_k^{s,\text{an}}$ is clearly continuous. Explicitely, given a polynomial $g = \sum_J g_J T^J \in k[T_1, \dots, T_s]$ and a point $z \in \mathbb{A}_k^{r,\text{an}}$,

$$|g(P_\alpha(z))| = \left| \left(\sum_J g_J \prod_{l=1}^s \left(\sum_{|I| \leq \delta} \chi_\alpha(a_{l,I}) T^I \right)^{j_l} \right) \sigma_{K/k}(z) \right| \quad (3.4)$$

To emphasize the fact that $\bar{\mathbb{D}}_k^{(\delta+1)rs}$ parametrizes analytic maps, we shall denote it from now on by $\text{Mor}_\delta^{r,s}$. For r, s fixed, we have constructed a map

$$\begin{aligned} \text{Ev} : \text{Mor}_\delta^{r,s} &\rightarrow \mathcal{C}^0(\mathbb{A}^{r,\text{an}}, \mathbb{A}^{s,\text{an}}) \\ \alpha &\mapsto \text{Ev}(\alpha) = P_\alpha. \end{aligned}$$

Note that for every fixed $\alpha \in \text{Mor}_\delta^{r,s}$, the map $z \mapsto \text{Ev}(\alpha)(z) = P_\alpha(z)$ can be expressed as

$$P_\alpha(z) = \pi_{K/k} \circ F_\alpha \circ \sigma_{K/k}(z),$$

where $F_\alpha : \mathbb{A}_K^{r,\text{an}} \rightarrow \mathbb{A}_K^{s,\text{an}}$ is the polynomial map

$$F_\alpha = (F_1, \dots, F_s) = \left(\sum_{|I| \leq \delta} \chi_\alpha(a_{1,I}) T^I, \dots, \sum_{|I| \leq \delta} \chi_\alpha(a_{s,I}) T^I \right)$$

Observe that the coefficients of F_α define a rigid point

$$\beta := ((\chi_\alpha(a_{1,I}))_{|I| \leq \delta}, \dots, (\chi_\alpha(a_{s,I}))_{|I| \leq \delta}) \in \text{Mor}_{\delta,K}^{r,s}$$

and $\pi_{K/k}(\beta) = \alpha$.

Proposition 3.3. *Let r, s, δ be three positive integers and let α be a point in the Berkovich analytic space $\text{Mor}_\delta^{r,s} := \bar{\mathbb{D}}_k^{(\delta+1)rs}$.*

Then for every fixed $z \in \mathbb{A}_k^{r,\text{an}}$ and every sequence of points α_n in $\text{Mor}_\delta^{r,s}$ converging to $\alpha \in \text{Mor}_\delta^{r,s}$, one has $P_{\alpha_n}(z) \rightarrow P_\alpha(z)$.

Moreover, if $\beta = ((b_{1,I})_{|I| \leq \delta}, \dots, (b_{s,I})_{|I| \leq \delta})$ is any rigid point in $\text{Mor}_{\delta,\mathcal{H}(\alpha)}^{r,s}$ such that $\pi_{\mathcal{H}(\alpha)/k}(\beta) = \alpha$, then the $\mathcal{H}(\alpha)$ -analytic map $F_\beta = (F_1, \dots, F_s)$, where $F_l = \sum_{|I| \leq \delta} b_{l,I} T^I$ for $l = 1, \dots, s$, satisfies

$$P_\alpha = \pi_{\mathcal{H}(\alpha)/k} \circ F_\beta \circ \sigma_{\mathcal{H}(\alpha)/k}.$$

Corollary 3.4. *Every sequence of polynomial maps $f_n : \mathbb{A}^{r,\text{an}} \rightarrow \mathbb{A}^{s,\text{an}}$ of uniformly bounded degree such that $f_n(\bar{\mathbb{D}}^r) \subseteq \bar{\mathbb{D}}^s$ for all n admits a pointwise converging subsequence whose limit $f : \mathbb{A}^{r,\text{an}} \rightarrow \mathbb{A}^{s,\text{an}}$ is continuous.*

Proof. The condition $f_n(\bar{\mathbb{D}}^r) \subseteq \bar{\mathbb{D}}^s$ implies that each f_n is given by a rigid point $\alpha_n \in \bar{\mathbb{D}}^{(\delta+1)rs}$. The result follows directly from Proposition 3.3 and [Poi13]. \square

Remark 3.5. *The function $(\alpha, z) \mapsto P_\alpha(z)$ does not define a continuous map on $|\mathbb{A}^{r,\text{an}}| \times |\text{Mor}_\delta^{r,s}|$. This phenomenon already appears when $r = s = \delta = 1$. Indeed, suppose by contradiction that there exists a continuous map $\varphi : |\mathbb{A}^{1,\text{an}}| \times |\bar{\mathbb{D}}| \rightarrow |\mathbb{A}^{1,\text{an}}|$ such that $\varphi(z, w) = z + w$ for any $z, w \in k$ with $|w| \leq 1$. Pick any sequence of points $\zeta_n \in k$ such that $|\zeta_n| = 1$ and $|\zeta_n - \zeta_m| = 1$ for $n \neq m$. Then ζ_n converges to the Gauss point x_g , and we have $\varphi(\zeta_n, -\zeta_n) = 0$ for all n , whereas*

$$\lim_n \varphi(\zeta_n, -\zeta_n) = \lim_n \varphi(\zeta_n, \zeta_n) = \varphi(x_g, x_g) = x_g.$$

This gives a contradiction.

Remark 3.6. *In general, the map*

$$\begin{aligned} \text{Ev} : \text{Mor}_\delta^{r,s} &\rightarrow \mathcal{C}^0(\mathbb{A}^{r,\text{an}}, \mathbb{A}^{s,\text{an}}) \\ \alpha &\mapsto \text{Ev}(\alpha) = P_\alpha \end{aligned}$$

is not injective. This phenomenon already occurs for $r = s = \delta = 1$.

Indeed, let $r = s = \delta = 1$. Denote by p_0 and p_1 the first and second projections $\text{Mor}_1^{1,1} \rightarrow \text{Mor}_0^{1,1}$. As above, denote by $k\{a_0, a_1\}$ the underlying affinoid algebra of $\text{Mor}_1^{1,1}$. Let $\alpha \in \text{Mor}_1^{1,1}$ be such that $p_0(\alpha) = x_g$. Now,

the point α can be viewed as a point in the fibre $p_0^{-1}(x_g) \simeq \bar{\mathbb{D}}_{\mathcal{H}(x_g)}$. Suppose further that α is a rigid point in $\bar{\mathbb{D}}_{\mathcal{H}(x_g)}$ corresponding to $q_0 + q_1 a_0 + q_2 a_0^2 \in k(a_0) \subset \mathcal{H}(x_g)$.

Fix a rigid point $z \in \mathbb{A}^{1,\text{an}}$ and pick some $g = \sum_j g_j T^j \in k[T]$. Then,

$$|g(P_\alpha(z))| = |g(P^{[z]}(\alpha))| = \left| \left(\sum_j g_j (a_0 + a_1 z)^j \right) (\alpha) \right|.$$

As $z \in k$, for suitable $g_{i,j} \in k$ we may write $\sum_j g_j (a_0 + a_1 z)^j = \sum_{i,j} g_{i,j} a_0^i a_1^j$. Then,

$$|g(P_\alpha(z))| = \left| \left(\sum_{i,j} g_{i,j} a_0^i a_1^j \right) (\alpha) \right| = \left| \left(\sum_j g_{i,j} a_0^i (q_0 + q_1 a_0 + q_2 a_0^2)^j \right) (x_g) \right|.$$

The image $P_\alpha(z)$ is not rigid in general. In order to compute it, approximate α by a the sequence of rigid points $\alpha_n = (\zeta_n, q_0 + q_1 \zeta_n + q_2 \zeta_n^2) \in \text{Mor}_1^{1,1}$, where $\zeta_n \in k$ are such that $|\zeta_n| = 1$ and $|\zeta_n - \zeta_m| = 1$ for all $n \neq m$. For $z \in \mathbb{A}^{1,\text{an}}$ rigid, $P_{\alpha_n}(z) = \zeta_n + (q_0 + q_1 \zeta_n + q_2 \zeta_n^2)z$. By Proposition 3.3, $P_{\alpha_n}(z) \rightarrow P_\alpha(z)$ as $n \rightarrow \infty$. The rigid point $z \in \mathbb{A}^{r,\text{an}}$ is thus mapped by P_α to the point in $\mathbb{A}^{1,\text{an}}$ corresponding to the closed ball

$$\bar{D}(zq_0; \max\{|1 + q_1 z|, |q_2 z|\}).$$

Let α' be a rigid point in $\bar{\mathbb{D}}_{\mathcal{H}(x_g)}$ corresponding to $q_0 + q_1 a_0 + q'_2 a_0^2 \in k(a_0) \subset \mathcal{H}(x_g)$, with $|q'_2| = |q_2|$. An analogous computation shows that any rigid point $z \in \mathbb{A}^{1,\text{an}}$ is sent by $P_{\alpha'}$ to the point in $\mathbb{A}^{1,\text{an}}$ corresponding to the closed ball

$$\bar{D}(zq_0; \max\{|1 + q_1 z|, |q'_2 z|\}).$$

It follows that P_α and $P_{\alpha'}$ agree on the set of rigid points and so they are equal.

Proof of Proposition 3.3. Fix a point $\alpha \in \text{Mor}_{\delta}^{r,s}$ and set $K = \mathcal{H}(\alpha)$.

Let us first show that P_α does not depend on the lift. To this end, pick $\beta_1 \neq \beta_2$ two rigid points in $\text{Mor}_{\delta,K}^{r,s}$ whose images by $\pi_{K/k}$ are equal to α . Denote by F_1 and F_2 respectively the K -analytic polynomial maps they induce on $\mathbb{A}_K^{r,\text{an}}$, and set $P_1 = \pi_{K/k} \circ F_1 \circ \sigma_{K/k}$ and $P_2 = \pi_{K/k} \circ F_2 \circ \sigma_{K/k}$.

By density, it suffices to check that P_1 and P_2 agree on the set of rigid points. Thus, let $z \in \mathbb{A}_k^{r,\text{an}}$ be rigid and pick $g = \sum_J g_J T^J \in k[T_1, \dots, T_s]$. Write $\beta_1 = (b_1, \dots, b_s)$, where $b_l = (b_{l,I})_{|I| \leq \delta} \in K^{(\delta+1)r}$ for $l = 1, \dots, s$. Then:

$$\begin{aligned} |g(P_1(z))| &= |g(\pi \circ F_1 \circ \sigma_{K/k}(z))| = |g(F_1(z))|_K \\ &= \left| \sum_J g_J \prod_{l=1}^s \left(\sum_{|I| \leq \delta} b_{l,I} z^I \right)^{j_l} \right|_K = |R(\beta_1)|_K, \end{aligned}$$

where R denotes the polynomial in $(\delta + 1)rs$ -variables with coefficients in k defined in (3.2). It follows that

$$|g(P_1(z))| = |R(\beta_1)| = |R(\pi_{K/k}(\beta_1))| = |R(\pi_{K/k}(\beta_2))| = |g(P_2(z))|.$$

Let us now prove the continuity statement. Fix a point $z \in \mathbb{A}_k^{r,\text{an}}$. Our aim is to construct a continuous map $P^{[z]} : \text{Mor}_\delta^{r,s} \rightarrow \mathbb{A}_k^{s,\text{an}}$ such that for all $\alpha \in \mathbb{D}_k^{(\delta+1)rs}$ we have $P^{[z]}(\alpha) = P_\alpha(z)$. To do so, consider the second projection $\pi_2 : \text{Mor}_\delta^{r,s} \times \mathbb{A}_k^{r,\text{an}} \rightarrow \mathbb{A}_k^{r,\text{an}}$. As above, the fibre over z is isomorphic to $\text{Mor}_{\delta, \mathcal{H}(z)}^{r,s}$. Let $\chi_z : k[T_1, \dots, T_r] \rightarrow \mathcal{H}(z)$ be the character corresponding to the point z , and set $L := \mathcal{H}(z)$.

Let $\iota_L : \text{Mor}_{\delta,L}^{r,s} \rightarrow \text{Mor}_\delta^{r,s} \times \mathbb{A}_k^{r,\text{an}}$ be the continuous map given by

$$\begin{aligned} k[T_1, \dots, T_r] \{ (a_{1,I})_{|I| \leq \delta}, \dots, (a_{s,I})_{|I| \leq \delta} \} &\rightarrow L \{ (a_{1,I})_{|I| \leq \delta}, \dots, (a_{s,I})_{|I| \leq \delta} \} \\ T_i &\mapsto \chi_z(T_i) \\ a_{l,I} &\mapsto a_{l,I}. \end{aligned}$$

Pick some $g = \sum_{|I| \leq \delta} g_I T^I$ in $k \{ (a_{1,I})_{|I| \leq \delta}, \dots, (a_{s,I})_{|I| \leq \delta} \} \{ T_1, \dots, T_r \}$. Fix $z \in \mathbb{D}_k^r$ and $\alpha \in \text{Mor}_\delta^{r,s}$. Then $\iota_K \circ \sigma_{K/k}(z) = \iota_L \circ \sigma_{L/k}(\alpha)$. Indeed, denote by $\chi_z(T) = (\chi_z(T_1), \dots, \chi_z(T_r)) \in L^r$. Then,

$$\begin{aligned} |g(\iota_K \circ \sigma_{K/k}(z))| &= |(\sum_{|I| \leq \delta} \chi_\alpha(g_I) \cdot T^I)(\sigma_{K/k}(z))| \\ &= \max_{|I| \leq \delta} |\chi_\alpha(g_I)|_K \cdot |T^I(z)| = \max_{|I| \leq \delta} |g_I(\alpha)| \cdot |\chi_z(T)^I|_L \\ &= |(\sum_{|I| \leq \delta} g_I \cdot \chi_z(T)^I)(\sigma_{L/k}(\alpha))| = |g(\iota_L \circ \sigma_{L/k}(\alpha))| \quad (3.5) \end{aligned}$$

Let $\Phi : \text{Mor}_\delta^{r,s} \times \mathbb{A}_k^{r,\text{an}} \rightarrow \mathbb{A}_k^{s,\text{an}}$ be the map defined above and consider the continuous map $P^{[z]} : \text{Mor}_\delta^{r,s} \rightarrow \mathbb{A}_k^{s,\text{an}}$, where

$$P^{[z]} = \Phi \circ \iota_L \circ \sigma_{L/k}.$$

As a consequence of (3.5), for all fixed $z \in \mathbb{D}_k^r$ and $\alpha \in \text{Mor}_\delta^{r,s}$,

$$P^{[z]}(\alpha) = P_\alpha(z).$$

Now let α_n be a sequence in $\text{Mor}_\delta^{r,s}$ converging to a point α . Then, by continuity we have that for every $z \in \mathbb{A}_k^{r,\text{an}}$

$$\lim_n P_{\alpha_n}(z) = \lim_n P^{[z]}(\alpha_n) = P^{[z]}(\alpha) = P_\alpha(z).$$

□

3.4. Continuity of pointwise limits of analytic maps. In this section, we complete the proof of Theorem A. First we show:

Theorem 3.7. *Let X be a basic tube defined over k and let Y be a k -affinoid space.*

Suppose that $f_n : X \rightarrow Y$ is a sequence of analytic maps that converges pointwise to a map f . Then f is continuous.

Remark 3.8. It is crucial here to assume that X has no boundary. Indeed, as pointed out in [FKT12, §4.2], the sequence of analytic maps from the closed unit disk $\bar{\mathbb{D}}$ to itself $f_n(z) = z^{2^{n!}}$ converges pointwise everywhere, but the limit map f is not continuous. The Gauss point x_g is a fixed point of f , but f maps the whole of $\bar{\mathbb{D}}$ to 0.

Proof of Theorem 3.7. We may assume that $Y = \bar{\mathbb{D}}^s$.

Suppose first that $X = \mathbb{D}^r$. Pick any sequence $f_n : \mathbb{D}^r \rightarrow \bar{\mathbb{D}}^s$ of analytic maps converging to f . It suffices to see that for every $0 < \rho < 1$, the restriction of f to $\bar{\mathbb{D}}^r(\rho)$ is continuous.

Each f_n is determined by a power series $f_l^{(n)} = \sum_I a_{l,I}^{(n)} T^I$, $l = 1, \dots, s$, with $|a_{l,I}^{(n)}| \rho^{|I|} \xrightarrow{|I| \rightarrow \infty} 0$ for all $0 < \rho < 1$. For each integer $\delta \in \mathbb{N}$, we introduce the truncated maps

$$f_n^\delta = (f_{1,\delta}^{(n)}, \dots, f_{s,\delta}^{(n)}) = \left(\sum_{|I| \leq \delta} a_{1,I}^{(n)} T^I, \dots, \sum_{|I| \leq \delta} a_{s,I}^{(n)} T^I \right).$$

as above. Replacing f_n by a subsequence if necessary, we may furthermore assume that for every $\delta \geq 0$ the sequence of points $\alpha_{n,\delta}$ defined by (3.1) is converging to some $\alpha_\delta \in \bar{\mathbb{D}}_k^{(\delta+1)rs}$. It follows from Proposition 3.3 that f_n^δ converges pointwise to the continuous map P_{α_δ} .

Fix any positive real number $0 < \rho < 1$, and recall the estimate (3.3). For any $g \in \mathcal{T}_s$ and any $z \in \bar{\mathbb{D}}^r(\rho)$, we have $||g(f_n(z))| - |g(f_n^\delta(z))|| \leq \rho^{\delta+1}$, so that

$$||g(f(z))| - |g(P_{\alpha_\delta}(z))|| \leq \rho^{\delta+1} \quad (3.6)$$

by letting $n \rightarrow \infty$. This implies that P_{α_δ} converges pointwise everywhere on $\bar{\mathbb{D}}^r$ to f .

We now prove the continuity of f . Since polydisks are Fréchet-Urysohn spaces by [Poi13], it suffices to check the continuity of f on sequences. Let z_m be a sequence of points in $\bar{\mathbb{D}}^r(\rho)$ converging to some z , and pick any $g \in \mathcal{T}_s$. We need to show that $|g(f(z_m))|$ converges to $|g(f(z))|$. Pick a positive real number $\epsilon > 0$, and δ large enough such that $\rho^{\delta+1} \leq \epsilon$. Since P_{α_δ} is continuous, we have $||g(P_{\alpha_\delta}(z_m))| - |g(P_{\alpha_\delta}(z))|| \rightarrow 0$ as $m \rightarrow \infty$, hence

$$\begin{aligned} ||g(f(z_m))| - |g(f(z))|| &\leq \max \{ ||g(f(z_m))| - |g(P_{\alpha_\delta}(z_m))||, \\ &\quad ||g(P_{\alpha_\delta}(z_m))| - |g(P_{\alpha_\delta}(z))||, ||g(P_{\alpha_\delta}(z))| - |g(f(z))|| \} < \epsilon, \end{aligned}$$

for all m large enough. This concludes the proof of the theorem in the case where X is the open unit polydisk.

If X is a general basic tube, we may write it as an increasing union of affinoids and use a diagonal extraction argument as in the proof of Theorem 3.1 to conclude. \square

Proof of Theorem A. Being σ -compact, X is the union of countably many compact sets K_n . Since it is a good analytic space without boundary, each compact set K_n is included in a finite union of open sets, each isomorphic to a basic tube. It follows that X is a countable union of basic tubes U_m .

By a diagonal extraction argument and Theorem 3.1, there exists a subsequence f_{n_k} converging pointwise on any open sets U_m , hence on X . By Theorem 3.7, the limit is continuous on every U_m and hence on X since they are open. \square

3.5. Convergence results over an arbitrary base field. We observe in this section that our main theorem also holds for any complete non-Archimedean valued base field k .

Theorem 3.9. *Let k be any complete non-Archimedean valued field. Let X be a good, reduced, σ -compact, boundaryless strictly k -analytic space and Y be a k -affinoid space.*

Then, every sequence of analytic maps $f_n : X \rightarrow Y$ has a pointwise converging subsequence whose limit map f is continuous.

Proof. Let K be the completed algebraic closure of k , and X_K, Y_K be the scalar extensions of X and Y respectively, see §2.6.

Pick a sequence $f_n : X \rightarrow Y$ of analytic maps and consider the analytic maps $F_n : X_K \rightarrow Y_K$ induced by base change. The following diagram commutes:

$$\begin{array}{ccc} X_K & \xrightarrow{F_n} & Y_K \\ \pi_{K/k} \downarrow & & \downarrow \pi_{K/k} \\ X & \xrightarrow{f_n} & Y \end{array}$$

Observe that the analytic space X_K is good and σ -compact, since the preimage $\pi_{K/k}^{-1}(U)$ of an affinoid domain U of X is an affinoid domain in X_K . It follows directly from the definition of the interior that X_K is boundaryless ([Ber90, Proposition 3.1.3]). Thus, by Theorem 3.7 we may assume that F_n is pointwise converging to a continuous map $F : X_K \rightarrow Y_K$. Pick a point $z \in X$. As $\pi_{K/k}$ is surjective, we may choose a point $z' \in \pi_{K/k}^{-1}(z)$. It follows that $f_n(z) = f_n(\pi_{K/k}(z')) = \pi_{K/k} \circ F_n(z')$, which tends to $\pi_{K/k} \circ F(z') := f(z)$ as n goes to infinity. The limit map f is well-defined. Indeed, if z', z'' are two points in $\pi_{K/k}^{-1}(z)$, then

$$\lim_n \pi_{K/k} \circ F_n(z') = \lim_n f_n(\pi_{K/k}(z')) = \lim_n f_n(\pi_{K/k}(z'')) = \lim_n \pi_{K/k} \circ F_n(z'').$$

It remains to check that f is continuous. Let A be any closed (hence compact) subset of Y . By continuity, the set $F^{-1}(\pi_{K/k}^{-1}(A))$ is closed. Recall

that the map $\pi_{K/k} : X_K \rightarrow X$ is proper. Since X_K and X are locally compact, then $\pi_{K/k}$ is closed. As a consequence, $f^{-1}(A) = \pi_{K/k} \left(F^{-1} \circ \pi_{K/k}^{-1}(A) \right)$ is closed. \square

4. ANALYTIC PROPERTIES OF POINTWISE LIMITS OF ANALYTIC MAPS

We analyse further the structure of the limit maps obtained in Theorem A, and prove they can be lifted to analytic maps after a suitable base change. To that end, we interpret analytic maps between an open and a closed polydisk as rigid points of the spectrum of a suitable Banach k -algebra.

Throughout this section, we fix two integers $r, s > 0$.

4.1. The infinite dimensional affinoid space $\text{Mor}_{\infty}^{r,s}$. Our aim is to build an infinite dimensional analytic space $\text{Mor}_{\infty}^{r,s}$ that parametrizes in a suitable sense the set of all analytic maps from \mathbb{D}_k^r to $\bar{\mathbb{D}}_k^s$.

Pick some $\delta \in \mathbb{N}$ and let $P : \mathbb{A}_k^{r,\text{an}} \rightarrow \mathbb{A}_k^{s,\text{an}}$ be a polynomial map of degree at most δ such that $P(\mathbb{D}_k^r) \subset \bar{\mathbb{D}}_k^s$. Since P can be written as

$$P(T_1, \dots, T_r) = \left(\sum_{|I| \leq \delta} a_{1,I} T^I, \dots, \sum_{|I| \leq \delta} a_{s,I} T^I \right)$$

with $|a_{l,I}| \leq 1$, the set of all such polynomial maps of degree at most δ can be endowed with a natural structure of affinoid space whose affinoid algebra is the Tate algebra $k\{a_{1,I}, \dots, a_{s,I}\}_{|I| \leq \delta} = k\{a_{l,I}\}_{|I| \leq \delta, 1 \leq l \leq s}$. We shall denote this space by $\text{Mor}_{\delta}^{r,s}$. It is isomorphic to the unit polydisk $\bar{\mathbb{D}}^{(\delta+1)rs}$.

Observe that for any given $\delta \in \mathbb{N}$ there exists a natural truncation map $\text{pr}_{\delta} : \text{Mor}_{\delta+1}^{r,s} \rightarrow \text{Mor}_{\delta}^{r,s}$, which is a surjective analytic map dual to the inclusion of Tate algebras $k\{a_{l,I}\}_{|I| \leq \delta, 1 \leq l \leq s} \subset k\{a_{l,I}\}_{|I| \leq \delta+1, 1 \leq l \leq s}$. These inclusions are isometric and we may so consider the inductive limit of this directed system. It is a normed k -algebra that we denote by $\mathcal{T}^{r,s}$.

In order to describe the elements of $\mathcal{T}^{r,s}$, we introduce the set \mathcal{S} of all maps $M : \{1, \dots, s\} \times \mathbb{N}^r \rightarrow \mathbb{N}$ having finite support and set $|M| = \sum_{l,I} M(l,I)$ for every $M \in \mathcal{S}$. We define \mathcal{S}_{δ} as the subset of \mathcal{S} consisting of all $M \in \mathcal{S}$ such that $M(l,I) = 0$ for all $|I| \geq \delta + 1$. Given $a = ((a_{1,I})_{|I| \leq \delta}, \dots, (a_{s,I})_{|I| \leq \delta})$ and $M \in \mathcal{S}$, we write

$$a^M = \prod_{1 \leq l \leq s, I \in \mathbb{N}^r} a_{l,I}^{M(l,I)}.$$

The k -algebra $\mathcal{T}^{r,s}$ consists of all power series that are of the form

$$\sum_{M \in \mathcal{S}_{\delta}} g_M \cdot a^M,$$

for some $\delta \in \mathbb{N}$ and whose coefficients $g_M \in k$ are such that $|g_M| \rightarrow 0$ as $|M| \rightarrow \infty$.

Remark 4.1. The k -algebra $\mathcal{T}^{r,s}$ is not complete. Take for instance $r = s = 1$ and consider the sequence $g_n = \sum_{i=1}^n \hat{g}_i \cdot a_{1,i} \in \mathcal{T}^{r,s}$. This is a Cauchy sequence as soon as the coefficients $\hat{g}_i \in k^*$ are such that $|\hat{g}_i| \rightarrow 0$ when $i \rightarrow \infty$, but it does not have any limit in $\mathcal{T}^{r,s}$.

The completion $\mathcal{T}_\infty^{r,s}$ of $\mathcal{T}^{r,s}$ is a Banach k -algebra consisting of all power series

$$\sum_{M \in \mathcal{S}} g_M \cdot a^M$$

such that for all $\epsilon > 0$ the set of $M \in \mathcal{S}$ such that $|g_M| > \epsilon$ is finite.

Definition 4.2. The space $\text{Mor}_\infty^{r,s}$ is the analytic spectrum of the Banach algebra $\mathcal{T}_\infty^{r,s}$.

For every $\delta \in \mathbb{N}$, the isometric inclusion $k\{a_{l,I}\}_{|I| \leq \delta, 1 \leq l \leq s} \subset \mathcal{T}_\infty^{r,s}$ defines a natural surjective continuous map $\text{Pr}_\delta^\infty : \text{Mor}_\infty^{r,s} \rightarrow \text{Mor}_\delta^{r,s}$. We may as well consider the inverse limit of all the spaces $\text{Mor}_\delta^{r,s}$, induced by the truncation maps $\text{pr}_\delta : \text{Mor}_{\delta+1}^{r,s} \rightarrow \text{Mor}_\delta^{r,s}$. These maps verify the equality $\text{pr}_\delta \circ \text{Pr}_{\delta+1}^\infty = \text{Pr}_\delta^\infty$ and induce a continuous map $\varphi : \text{Mor}_\infty^{r,s} \rightarrow \varprojlim_\delta \text{Mor}_\delta^{r,s}$.

We shall consider the inclusions $i_\delta : \text{Mor}_\delta^{r,s} \rightarrow \text{Mor}_\infty^{r,s}$ given by $\mathcal{T}_\infty^{r,s} \rightarrow k\{a_{l,I}\}_{|I| \leq \delta, 1 \leq l \leq s}$, sending $a_{l,I}$ to itself if $|I| \leq \delta$ and to 0 otherwise. These are closed immersions.

Proposition 4.3. The map $\varphi : \text{Mor}_\infty^{r,s} \rightarrow \varprojlim_\delta \text{Mor}_\delta^{r,s}$ is a homeomorphism. In particular, $\text{Mor}_\infty^{r,s}$ is compact.

Proof. The inverse limit $\varprojlim_\delta \text{Mor}_\delta^{r,s}$ is compact by Tychonoff, and $\text{Mor}_\infty^{r,s}$ is compact because it is the analytic spectrum of the k -banach algebra $\mathcal{T}_\infty^{r,s}$.

Let us show that $\varphi : \text{Mor}_\infty^{r,s} \rightarrow \varprojlim_\delta \text{Mor}_\delta^{r,s}$ is bijective.

Fix $\delta \geq 0$. Let $\pi_\delta : \varprojlim_\delta \text{Mor}_\delta^{r,s} \rightarrow \text{Mor}_\delta^{r,s}$ be the natural map and $\text{pr}_\delta : \text{Mor}_{\delta+1}^{r,s} \rightarrow \text{Mor}_\delta^{r,s}$ the truncation map. We know that $\text{Pr}_\delta^\infty = \pi_\delta \circ \varphi$. Pick a point $y \in \varprojlim_\delta \text{Mor}_\delta^{r,s}$ and consider $\pi_\delta(y) \in \text{Mor}_\delta^{r,s}$. Consider the set K_δ consisting of all the points $\alpha \in \text{Mor}_\infty^{r,s}$ such that $\text{Pr}_\delta^\infty(\alpha) = \pi_\delta(y)$. By surjectivity of the maps Pr_δ^∞ , the subset K_δ is non-empty. Clearly, we have that $K_{\delta+1} \subseteq K_\delta$. Every K_δ is compact and so the intersection $\bigcap_{\delta \geq 0} K_\delta$ is non-empty. This shows that φ is surjective.

For the injectivity, let α, α' be two points in $\text{Mor}_\infty^{r,s}$ having the same image in $\varprojlim_\delta \text{Mor}_\delta^{r,s}$. We have to check that $|g(\alpha)| = |g(\alpha')|$ for every $g \in \mathcal{T}_\infty^{r,s}$, that by density reduces to the case where $g \in \mathcal{T}^{r,s}$. We know that $\text{Pr}_\delta^\infty(\alpha) = \text{Pr}_\delta^\infty(\alpha') \in \text{Mor}_\delta^{r,s}$ for all δ . Given $g \in \mathcal{T}^{r,s}$ observe that it lies in $k\{a_{l,I}\}_{|I| \leq \delta, 1 \leq l \leq s}$ for some $\delta \geq 0$. Thus,

$$|g(\alpha)| = |g(\text{Pr}_\delta^\infty(\alpha))| = |g(\text{Pr}_\delta^\infty(\alpha'))| = |g(\alpha')|.$$

□

Recall from §2 the definition of the complete residue field $\mathcal{H}(\alpha)$ of a point $\alpha \in \text{Mor}_\infty^{r,s}$. To simplify notation, we write $\alpha_\delta = \text{Pr}_\delta^\infty(\alpha)$.

Proposition 4.4. *Let α be a point in $\text{Mor}_\infty^{r,s}$. For every $\delta \in \mathbb{N}$ the inclusion of Banach k -algebras $k\{a_{l,I}\}_{1 \leq l \leq s, |I| \leq \delta} \subset \mathcal{T}_\infty^{r,s}$ induces an extension of valued fields $\mathcal{H}(\alpha)/\mathcal{H}(\alpha_\delta)$.*

The complete residue field $\mathcal{H}(\alpha)$ is isomorphic to the completion of the inductive limit of valued fields $\varinjlim_\delta \mathcal{H}(\alpha_\delta)$.

Proof. A point $\alpha \in \text{Mor}_\infty^{r,s}$ corresponds to a seminorm on the k -algebra $\mathcal{T}_\infty^{r,s}$, whose restriction to $k\{a_{l,I}\}_{|I| \leq \delta, 1 \leq l \leq s}$ is the seminorm α_δ . The kernel of α_δ is the intersection of $k\{a_{l,I}\}_{|I| \leq \delta, 1 \leq l \leq s}$ with $\ker(\alpha)$. This induces inclusions

$$k\{a_{l,I}\}_{|I| \leq \delta, 1 \leq l \leq s} / \ker(\alpha_\delta) \subset \mathcal{T}_\infty^{r,s} / \ker(\alpha). \quad (4.1)$$

It follows that there are inclusions $\mathcal{H}(\alpha_\delta) \subset \mathcal{H}(\alpha)$, and thus the direct limit of the $\mathcal{H}(\alpha_\delta)$ is naturally contained in $\mathcal{H}(\alpha)$. In order to show that $\mathcal{H}(\alpha)$ is isomorphic to the completion of $\varinjlim_\delta \mathcal{H}(\alpha_\delta)$, it suffices to show that $\varinjlim_\delta \mathcal{H}(\alpha_\delta)$ is dense in $\mathcal{H}(\alpha)$.

Consider the field $K := \varinjlim_\delta \text{Frac}(k\{a_{l,I}\}_{|I| \leq \delta, 1 \leq l \leq s} / \ker(\alpha_\delta))$. Clearly, K is contained in $\varinjlim_\delta \mathcal{H}(\alpha_\delta)$. By (4.1) and by the definition of $\mathcal{T}_\infty^{r,s}$, we also know that K is dense in $\text{Frac}(\mathcal{T}_\infty^{r,s} / \ker(\alpha))$. The latter is by definition dense in $\mathcal{H}(\alpha)$, which proves that $\varinjlim_\delta \mathcal{H}(\alpha_\delta)$ is dense in $\mathcal{H}(\alpha)$. \square

4.2. Universal property of the space $\text{Mor}_\infty^{r,s}$. Let us specify in which sense $\text{Mor}_\infty^{r,s}$ parametrizes the space of analytic maps from \mathbb{D}_k^r to $\bar{\mathbb{D}}_k^s$. Recall from §2.2 that a morphism between the spectra of two Banach k -algebras is by definition a continuous map induced by a bounded morphism between the underlying algebras.

Theorem 4.5. *The association $(T_1, \dots, T_s) \mapsto (\sum a_{1,I} T^I, \dots, \sum a_{s,I} T^I)$ where I ranges over \mathbb{N}^r , induces a bounded morphism of Banach k -algebras*

$$k\{T_1, \dots, T_s\} \rightarrow \mathcal{T}_\infty\{\rho^{-1}T_1, \dots, \rho^{-1}T_r\}$$

for every $\rho < 1$.

The induced morphism $\Phi : \text{Mor}_\infty^{r,s} \times \mathbb{D}_k^r \rightarrow \bar{\mathbb{D}}_k^s$ satisfies the following universal property. For any strictly k -affinoid space X and for any analytic map $F : X \times \mathbb{D}_k^r \rightarrow \bar{\mathbb{D}}_k^s$ there exists a unique morphism $g : X \rightarrow \text{Mor}_\infty^{r,s}$ such that $F(x, z) = \Phi(g(x), z)$ for all $x \in X(k)$ and $z \in \mathbb{D}^r(k)$.

Proof. Let X be a strictly k -affinoid space with affinoid algebra \mathcal{A} . Let $F : X \times \mathbb{D}_k^r \rightarrow \bar{\mathbb{D}}_k^s$ be an analytic map, which is given by

$$(T_1, \dots, T_s) \mapsto \left(\sum b_{1,I} T^I, \dots, \sum b_{s,I} T^I \right),$$

where $b_{l,I} \in \mathcal{A}$ are such that $\sup_{l,I} |b_{l,I}(x)| \leq 1$ for all $x \in X$. Let $g : X \rightarrow \text{Mor}_\infty^{r,s}$ be the analytic map given by $a_{l,I} \mapsto b_{l,I}$ for all $I \in \mathbb{N}^r$ and all $1 \leq l \leq s$. Given a rigid point $x \in X$ together with a rigid point $z \in \mathbb{D}^r$, they define a rigid point in the product $X \times \mathbb{D}^r$ and by construction we have $F(x, z) = \Phi(g(x), z)$.

Conversely, let $h : X \rightarrow \text{Mor}_{\infty}^{r,s}$ be an analytic map sending $a_{l,I}$ to $c_{l,I} \in \mathcal{A}$ and satisfying $F(x, z) = \Phi(h(x), z)$ for all $x \in X(k)$ and all $z \in \mathbb{D}^r(k)$. For every fixed $x \in X(k)$, consider the analytic map $z \in \mathbb{D}^r \mapsto \Phi(h(x), z)$. By hypothesis, it agrees with the map $z \in \mathbb{D}^r \mapsto \Phi(g(x), z)$, and so $b_{l,I}(x) = c_{l,I}(x)$ for every $I \in \mathbb{N}^r$ and $1 \leq l \leq s$. As the equalities hold for every rigid $x \in X$, we conclude that $h = g$. \square

Recall that a point $\alpha \in \text{Mor}_{\infty}^{r,s}$ is rigid if and only if its complete residue field $\mathcal{H}(\alpha)$ is equal to k . When α is rigid, then $\Phi(\alpha, \cdot)$ defines an analytic map from \mathbb{D}^r to $\bar{\mathbb{D}}^s$. The previous theorem shows in particular that the set of analytic maps from \mathbb{D}^r to $\bar{\mathbb{D}}^s$ is in bijection with the set $\{\Phi(\alpha, \cdot) : \alpha \in \text{Mor}_{\infty}^{r,s}(k)\}$, hence with the set of rigid points in $\text{Mor}_{\infty}^{r,s}$.

The following theorem is a generalization of Proposition 3.3 to analytic maps.

Theorem 4.6. *There exists a map Ev from $\text{Mor}_{\infty}^{r,s}$ to the space of continuous functions $\mathcal{C}^0(\mathbb{D}^r, \bar{\mathbb{D}}^s)$ such that the following holds:*

- (1) *The map $\text{Ev}(\alpha)$ is analytic if and only if the point $\alpha \in \text{Mor}_{\infty}^{r,s}$ is rigid. In that case, it is of the form $\text{Ev}(\alpha) = \Phi(\alpha, \cdot)$.*
- (2) *For any fixed $z \in \mathbb{D}^r$ and for any sequence $\{\alpha^{(n)}\} \subset \text{Mor}_{\infty}^{r,s}$ converging to some $\alpha \in \text{Mor}_{\infty}^{r,s}$, we have $\text{Ev}(\alpha^{(n)})(z) \rightarrow \text{Ev}(\alpha)(z)$.*
- (3) *For every $z \in \mathbb{D}^r$ and for every $\alpha \in \text{Mor}_{\infty}^{r,s}$ we have $\text{Ev}(\alpha_{\delta})(z) \rightarrow \text{Ev}(\alpha)(z)$ as δ goes to infinity, where $\alpha_{\delta} := \text{Pr}_{\delta}^{\infty}(\alpha)$.*

Proof. The map $\text{Ev} : \text{Mor}_{\infty}^{r,s} \rightarrow \mathcal{C}^0(\mathbb{D}^r, \bar{\mathbb{D}}^s)$ is given as follows. Fix a point $\alpha \in \text{Mor}_{\infty}^{r,s}$ and consider the first projection $\pi_1 : \text{Mor}_{\infty}^{r,s} \times \mathbb{D}_k^r \rightarrow \text{Mor}_{\infty}^{r,s}$. The fibre $\pi_1^{-1}(\alpha)$ is isomorphic to $\mathbb{D}_{\mathcal{H}(\alpha)}^r$ (cf. §2.2). We can thus consider the inclusion map $\iota_{\mathcal{H}(\alpha)} : \mathbb{D}_{\mathcal{H}(\alpha)}^r \rightarrow \text{Mor}_{\infty}^{r,s} \times \mathbb{D}_k^r$, given by

$$\begin{aligned} \mathcal{T}_{\infty}^{r,s} \{\rho^{-1}T_1, \dots, \rho^{-1}T_r\} &\rightarrow \mathcal{H}(\alpha) \{\rho^{-1}T_1, \dots, \rho^{-1}T_r\} \\ T_i &\mapsto T_i \\ a_{l,I} &\mapsto \chi_{\alpha}(a_{l,I}) \end{aligned} \tag{4.2}$$

for $\rho < 1$, where $\chi_{\alpha} : \mathcal{T}_{\infty}^{r,s} \rightarrow \mathcal{H}(\alpha)$ denotes the character associated to the point α . Let $\sigma_{\mathcal{H}(\alpha)/k} : \mathbb{D}_k^r \rightarrow \mathbb{D}_{\mathcal{H}(\alpha)}^r$ be the continuous map discussed in §2.6. Let Φ be the analytic map from Theorem 4.5. We set:

$$\text{Ev}(\alpha) = \Phi \circ \iota_{\mathcal{H}(\alpha)} \circ \sigma_{\mathcal{H}(\alpha)/k}.$$

Clearly, $\text{Ev}(\alpha)$ is a continuous map from \mathbb{D}_k^r to $\bar{\mathbb{D}}_k^s$. Specifically, for any $z \in \mathbb{D}^r$ and for any $g = \sum_J g_J T^J$ in $k\{T_1, \dots, T_s\}$, we have

$$|g(\text{Ev}(\alpha)(z))| = \left| \sum_J g_J \prod_{l=1}^s \left(\sum_I \chi_{\alpha}(a_{l,I}) \cdot T^I \right)^{j_l} (\sigma_{\mathcal{H}(\alpha)/k}(z)) \right|. \tag{4.3}$$

- (1) The point $\alpha \in \text{Mor}_{\infty}^{r,s}$ is rigid if and only if $\mathcal{H}(\alpha) = k$. In this situation, $\iota_{\mathcal{H}(\alpha)}$ is in fact an analytic map between k -analytic spaces and $\sigma_{\mathcal{H}(\alpha)/k}$ is the identity on \mathbb{D}_k^r . If α is not rigid, then $\chi_{\alpha}(a_{l,I})$ does

not belong to k for some $I \in \mathbb{N}^r$ and for some $1 \leq l \leq s$. It follows from (4.3) that the $\text{Ev}(\alpha)$ is not defined over k .

- In addition, observe that if α is rigid then the fibre $\pi_1^{-1}(\alpha)$ is homeomorphic to \mathbb{D}_k^r . Then, for every $z \in \mathbb{D}_k^r$ the data (α, z) define a point in $\text{Mor}_\infty^{r,s} \times \mathbb{D}_k^r$, and so $\iota_k(z) = (a, z)$. Thus, $\text{Ev}(\alpha) = \Phi(\alpha, \cdot)$.
- (2) Fix a point $z \in \mathbb{D}_k^r$ and consider the projection $\pi_2 : \text{Mor}_\infty^{r,s} \times \mathbb{D}_k^r \rightarrow \mathbb{D}_k^r$. The fibre $\pi_2^{-1}(z)$ is isomorphic to $\text{Mor}_{\infty, \mathcal{H}(z)}^{r,s}$. Consider the inclusion $\iota_{\mathcal{H}(z)} : \text{Mor}_{\infty, \mathcal{H}(z)}^{r,s} \rightarrow \text{Mor}_\infty^{r,s} \times \mathbb{D}_k^r$, given by

$$\begin{aligned} \mathcal{T}_\infty^{r,s} \{\rho^{-1}T_1, \dots, \rho^{-1}T_r\} &\rightarrow \mathcal{T}_\infty^{r,s} \hat{\otimes}_k \mathcal{H}(z) \\ T_i &\mapsto \chi_z(T_i) \\ a_{l,I} &\mapsto a_{l,I} \end{aligned}$$

for $\rho < 1$, where $\chi_z : k\{\rho^{-1}T_1, \dots, \rho^{-1}T_r\} \rightarrow \mathcal{H}(z)$ denotes the character associated to the point z . A computation analogous to (3.5) shows that for all fixed $z \in \mathbb{D}_k^r$ and $\alpha \in \text{Mor}_\infty^{r,s}$,

$$\iota_{\mathcal{H}(\alpha)} \circ \sigma_{\mathcal{H}(\alpha)/k}(z) = \iota_{\mathcal{H}(z)} \circ \sigma_{\mathcal{H}(z)/k}(\alpha).$$

Consider the continuous map $\Psi(z) : \text{Mor}_\infty^{r,s} \rightarrow \bar{\mathbb{D}}_k^s$, defined as the composite $\Psi(z) = \Phi \circ \iota_{\mathcal{H}(z)} \circ \sigma_{\mathcal{H}(z)/k}$. For every fixed $\alpha \in \text{Mor}_\infty^{r,s}$ and every fixed $z \in \mathbb{D}^r$, we have

$$\Psi(z)(\alpha) = \text{Ev}(\alpha)(z).$$

If $\alpha^{(n)}$ is a sequence of points in $\text{Mor}_\infty^{r,s}$ converging to α , then the continuity of $\Psi(z)$ implies that $\Psi(z)(\alpha^{(n)})$ converges to $\Psi(z)(\alpha)$ as n goes to infinity, and so

$$\text{Ev}(\alpha^{(n)})(z) \xrightarrow{n \rightarrow \infty} \text{Ev}(\alpha)(z).$$

- (3) Fix $\alpha \in \text{Mor}_\infty^{r,s}$ and set $\alpha_\delta := \text{Pr}_\delta^\infty(\alpha) \in \text{Mor}_\delta^{r,s}$. For every δ , consider the inclusion $i_\delta : \text{Mor}_\delta^{r,s} \rightarrow \text{Mor}_\infty^{r,s}$.

By the previous item it suffices to prove that $i_\delta(\alpha_\delta) \rightarrow \alpha$ in $\text{Mor}_\infty^{r,s}$ as δ tends to infinity. Pick $g = \sum_{M \in \mathcal{S}} g_M \cdot a^M \in \mathcal{T}_\infty^{r,s}$ and compute:

$$\begin{aligned} |g(i_\delta(\alpha_\delta)) - g(\alpha)| &= \left| \left(\sum_{M \in \mathcal{S}_\delta} g_M \cdot a^M \right)(\alpha_\delta) - \left(\sum_{M \in \mathcal{S}} g_M \cdot a^M \right)(\alpha) \right| \\ &= \left| \left(\sum_{M \in \mathcal{S} \setminus \mathcal{S}_\delta} g_M \cdot a^M \right)(\alpha) \right| \leq \max_{M \in \mathcal{S} \setminus \mathcal{S}_\delta} |g_M|, \end{aligned}$$

which tends to 0 as δ goes to infinity.

□

4.3. The space $\text{Mor}_\infty^{r,s}$ is Fréchet-Urysohn. We prove a technical result that is a key step in the proof of Theorem B.

Theorem 4.7. *The space $\text{Mor}_\infty^{r,s}$ is Fréchet-Urysohn.*

We follow Poineau's proof of the fact that analytic spaces are Fréchet-Urysohn [Poi13, Proposition 5.2], which relies on [Poi13, Théorème 4.22].

Recall that the *Shilov boundary* of the analytic spectrum of a k -Banach algebra \mathcal{A} is the smallest closed subset of $\mathcal{M}(\mathcal{A})$ where every $g \in \mathcal{A}$ reaches its maximum. It is a non-empty closed subset of $\mathcal{M}(\mathcal{A})$. In the following we deal with subfields l of k that are of *countable type* over the prime subfield k_p of k , in the sense of [BGR84]. Observe that this amounts for l to have a dense countable subset.

The following proposition is an infinite dimensional analogue of [Poi13, Théorème 4.22].

Proposition 4.8. *For every point α in $\text{Mor}_\infty^{r,s}$ there exists a subfield l of k that is of countable type over the prime subfield k_p of k and satisfying the following property. Let l' be any subfield of k with $l \subset l' \subset k$ and let $\pi_{k/l'}^\infty : \text{Mor}_{\infty,k}^{r,s} \rightarrow \text{Mor}_{\infty,l'}^{r,s}$ be the base change morphism. Then α is the unique point in the Shilov boundary of the fibre $(\pi_{k/l'}^\infty)^{-1}(\pi_{k/l'}^\infty(\alpha))$.*

Proof of Proposition 4.8. The space $\text{Mor}_\infty^{r,s}$ is the projective limit of $\text{Mor}_{\delta,k}^{r,s}$ with the morphisms $\text{Pr}_{\delta,k}^\infty : \text{Mor}_{\infty,k}^{r,s} \rightarrow \text{Mor}_{\delta,k}^{r,s}$ for $\delta \in \mathbb{N}^*$ (cf. Theorem 4.3). A point α in $\text{Mor}_{\infty,k}^{r,s}$ is thus determined by a sequence $(\alpha_\delta)_{\delta \geq 0}$, where each α_δ lies in $\text{Mor}_{\delta,k}^{r,s}$ and satisfies $\text{pr}_{\delta+1}(\alpha_{\delta+1}) = \alpha_\delta$ for the projections $\text{pr}_{\delta+1} : \text{Mor}_{\delta+1,k}^{r,s} \rightarrow \text{Mor}_{\delta,k}^{r,s}$.

To every α_δ we apply [Poi13, Théorème 4.22]. We obtain a field $l^\delta \subset k$ that is of countable type over the prime subfield k_p of k and such that for any subfield $l \subset l' \subset k$ the point α_δ is the only point in the Shilov boundary of $(\pi_{k/l'}^\delta)^{-1}(\pi_{k/l'}^\delta(\alpha_\delta))$, where $\pi_{k/l'}^\delta : \text{Mor}_{\delta,k}^{r,s} \rightarrow \text{Mor}_{\delta,l'}^{r,s}$ denotes the base change morphism.

Let l be the smallest field containing all the l^δ . By construction, l is contained in k and is of countable type over k_p . We may assume in addition that l is algebraically closed.

The equality $\pi_{k/l'}^\delta \circ \text{Pr}_{\delta,k}^\infty = \text{Pr}_{\delta,l'}^\infty \circ \pi_{k/l'}^\infty$ implies that $\text{Pr}_{\delta,k}^\infty$ maps the fibre $(\pi_{k/l'}^\infty)^{-1}(\pi_{k/l'}^\infty(\alpha))$ to the fibre $(\pi_{k/l'}^\delta)^{-1}(\pi_{k/l'}^\delta(\alpha_\delta))$. We show that α belongs to the Shilov boundary of $(\pi_{k/l'}^\infty)^{-1}(\pi_{k/l'}^\infty(\alpha))$. Pick an element $g \in \mathcal{T}_\infty^{r,s}$. As $\mathcal{T}^{r,s}$ is dense in $\mathcal{T}_\infty^{r,s}$, we may assume that g lies in $k\{a_{l,I}\}_{|I| \leq \delta, 1 \leq l \leq s}$ for some $\delta \geq 0$. Thus, $|g(\alpha)| = |g(\alpha_\delta)|$, which is the maximum value of g , since α_δ belongs to the Shilov boundary of $(\pi_{k/l'}^\delta)^{-1}(\pi_{k/l'}^\delta(\alpha_\delta))$.

Pick a point $\beta \in (\pi_{k/l'}^\infty)^{-1}(\pi_{k/l'}^\infty(\alpha))$ different from α , i.e. such that $\beta_\delta \neq \alpha_\delta$ for some $\delta \geq 0$. If $g \in k\{a_{l,I}\}_{|I| \leq \delta}$, we see that

$$|g(\beta)| = |g(\beta_\delta)| < |g(\alpha_\delta)| = |g(\alpha)|,$$

showing that α is the unique point in the Shilov boundary of the space $(\pi_{k/l'}^\infty)^{-1}(\pi_{k/l'}^\infty(\alpha))$. \square

Proof of Theorem 4.7. Let A be any subset of $\text{Mor}_\infty^{r,s}$ and let α be a point in the closure of A . Let l be the subfield of k associated to α from Proposition 4.8. Let $l \subset l' \subset k$ be any subfield of k that is of countable type over l . Every polydisk $\text{Mor}_{\delta,l'}^{r,s}$ is first countable, and as a consequence so is the countable product of all the $\text{Mor}_{\delta,l'}^{r,s}$. The space $\text{Mor}_\infty^{r,s}$ is a subspace of the product $\prod_\delta \text{Mor}_{\delta,l'}^{r,s}$ by Proposition 4.3, and thus is first countable.

Copying Poineau's proof of [Poi13, Proposition 5.2] and using Proposition 4.8, we may find a sequence of points $\alpha^{(n)}$ in A converging to α . \square

4.4. Lifting properties of limit maps. Continuous maps of the form $\text{Ev}(\alpha) : \mathbb{D}^r \rightarrow \bar{\mathbb{D}}^s$ are very special, as they exhibit properties that are distinctive of analytic functions. We shall prove that they lift to analytic maps after a suitable base change and that the graph of $\text{Ev}(\alpha)$ is well-defined in the analytic product $\mathbb{D}^r \times \mathbb{D}^s$ and not just in the topological product $|\mathbb{D}^r| \times |\mathbb{D}^s|$.

Recall from §2.6 the definition of the continuous map $\sigma_{K/k} : X \rightarrow X_K$.

Theorem 4.9. *Let α be a point in $\text{Mor}_\infty^{r,s}$. Then there exists a closed subset Γ_α of $\mathbb{D}^r \times \bar{\mathbb{D}}^s$ such that the first projection $\pi_1 : \Gamma_\alpha \rightarrow \mathbb{D}^r$ is a homeomorphism and such that for every $z \in \mathbb{D}^r$ the image under of $\Gamma_\alpha \cap \pi_1^{-1}(z)$ under the second projection is equal to $\text{Ev}(\alpha)(z) \in \bar{\mathbb{D}}^s$.*

Moreover, there exist a complete extension K/k and a K -analytic map $F_\alpha : \mathbb{D}_K^r \rightarrow \bar{\mathbb{D}}_K^s$ such that $\text{Ev}(\alpha) = \pi_{K/k} \circ F_\alpha \circ \sigma_{K/k}$.

Proof. Fix α in $\text{Mor}_\infty^{r,s}$ and denote by $\mathcal{H}(\alpha)$ its complete residue field. We define Γ_α as the image of a continuous map $\varphi : \mathbb{D}^r \rightarrow \mathbb{D}^r \times \bar{\mathbb{D}}^s$, that we construct as follows.

Let $\iota_{\mathcal{H}(\alpha)} : \mathbb{D}_{\mathcal{H}(\alpha)}^r \rightarrow \text{Mor}_\infty^{r,s} \times \mathbb{D}_k^r$ be the inclusion map defined in (4.2). Let $\Upsilon : \text{Mor}_\infty^{r,s} \times \mathbb{D}_k^r \rightarrow \mathbb{D}_k^r \times \bar{\mathbb{D}}_k^s$ be given by

$$\begin{aligned} k\{\rho^{-1}T_1, \dots, \rho^{-1}T_r\}\{S_1, \dots, S_s\} &\rightarrow \mathcal{T}_\infty^{r,s}\{\rho^{-1}T_1, \dots, \rho^{-1}T_r\} \\ T_i &\mapsto T_i \\ S_l &\mapsto \sum_I a_{l,I} T^I. \end{aligned}$$

Let $\sigma_{\mathcal{H}(\alpha)/k} : \mathbb{D}_k^r \rightarrow \mathbb{D}_{\mathcal{H}(\alpha)}^r$. We set $\varphi = \Upsilon \circ \iota_{\mathcal{H}(\alpha)} \circ \sigma_{\mathcal{H}(\alpha)/k}$. Explicitly, for any $z \in \mathbb{D}^r$ and every $g = \sum_J g_J S^J$, where $J = (j_1, \dots, j_s)$ and $g_J \in k\{\rho^{-1}T_1, \dots, \rho^{-1}T_r\}$ are such that $|g_J| \rightarrow 0$ as $|J| \rightarrow 0$, we have:

$$|g(\varphi(z))| = \left| \sum_J g_J \prod_{l=1}^s \left(\sum_I \chi_\alpha(a_{l,I}) \cdot T^I \right)^{j_l} (\sigma_{\mathcal{H}(\alpha)/k}(z)) \right|. \quad (4.4)$$

Consider the projections π_1 and π_2 on $\mathbb{D}^r \times \bar{\mathbb{D}}^s$ to the first and second factor respectively. It is an immediate consequence of the previous computation

and (4.3) that

$$\pi_2(\varphi(z)) = \text{Ev}(\alpha)(z).$$

If no variables S_l appear in the expression of g , then g lies in the algebra $k\{\rho^{-1}T_1, \dots, \rho^{-1}T_r\}$. Thus, by (4.4) we see that $|g(\varphi(z))| = |g(z)|$, and so

$$\pi_1(\varphi(z)) = z.$$

It remains to check that the image Γ_α of φ is a closed subset of $\mathbb{D}^r \times \bar{\mathbb{D}}^s$. Let z_n be a sequence of points in \mathbb{D}^r such that $\varphi(z_n)$ converges to some point x in $\mathbb{D}^r \times \bar{\mathbb{D}}^s$. As $\pi_1(\varphi(z_n)) = z_n$, we see that z_n converges to $\pi_1(x) \in \mathbb{D}^r$, and by continuity of φ we have that $x = \varphi(\pi_1(x))$ lies in Γ_α .

Let K be the complete residue field $\mathcal{H}(\alpha)$. Consider the $\mathcal{H}(\alpha)$ -analytic map

$$F_\alpha = \left(\sum_I \chi_\alpha(a_{1,I}) \cdot T^I, \dots, \sum_I \chi_\alpha(a_{s,I}) \cdot T^I \right),$$

where I ranges over \mathbb{N}^r . A direct computation together with (4.3) show that $\text{Ev}(\alpha) = \pi_{\mathcal{H}(\alpha)/k} \circ F_\alpha \circ \sigma_{\mathcal{H}(\alpha)/k}$. \square

4.5. Proof of Theorem B. We may always assume $Y = \bar{\mathbb{D}}_k^s$. Suppose first that $X = \mathbb{D}_k^r$. Each analytic map f_n is of the form $f_n = \text{Ev}(\alpha^{(n)})$ for some rigid point $\alpha^{(n)} \in \text{Mor}_\infty^{r,s}$ by Theorem 4.6. It was shown in Proposition 4.3 that the space $\text{Mor}_\infty^{r,s}$ is Fréchet-Urysohn so that we may assume that $\alpha^{(n)}$ converges to some point $\alpha \in \text{Mor}_\infty^{r,s}$. The limit map f is precisely $\text{Ev}(\alpha)$ (cf. Theorem 4.6) and we conclude by Theorem 4.9.

Let now X be any good, reduced, σ -compact k -analytic space. Pick a point $x \in X$ and an affinoid neighbourhood Z of x containing x in its interior. Fix a distinguished closed immersion of Z into some closed unit polydisk $\bar{\mathbb{D}}^r$. For every n we may find an analytic map $\hat{f}_n : \bar{\mathbb{D}}^r \rightarrow \bar{\mathbb{D}}^s$ such that $\hat{f}_n|_Z = f_n$ applying the same argument as in the proof of Theorem 3.1. We are thus reduced to the previous case, and this concludes the proof of Theorem B. \square

5. WEAKLY ANALYTIC MAPS

In this section we look more precisely at the properties of continuous limits of analytic functions. As before, k is any complete valued field which is algebraically closed.

5.1. Definition and first properties. We begin with a definition.

Definition 5.1. *Let X and Y be any two good k -analytic spaces, and let $f : X \rightarrow Y$ be a continuous map.*

We say that f is weakly analytic if for every point $x \in X$ there exist an affinoid neighbourhood U of x , a complete field extension K/k and an analytic map $F : U_K \rightarrow Y_K$ such that $f|_U = \pi_{K/k} \circ F \circ \sigma_{K/k}$.

It will be convenient to denote by $\text{WA}(X, Y)$ the set of all weakly analytic maps from X to Y .

Clearly, the set $\text{Mor}(X, Y)$ of analytic maps from X to Y is a subset of $\text{WA}(X, Y)$. It is also a strict subset since any constant map is weakly analytic, but it is analytic only if the constant is a rigid point.

Proposition 5.2. *Weakly analytic maps are stable under composition.*

Proof. Since being weakly analytic is a local notion, it suffices to argue in the affinoid case.

Let X , Y and Z be k -affinoid spaces. Consider $f \in \text{WA}(X, Y)$ and $g \in \text{WA}(Y, Z)$. Pick a point $x \in X$. Let U be an affinoid neighbourhood of x and let K be an extension of k such that $f|_U = \pi_{K/k} \circ F \circ \sigma_{K/k}$ for some K -analytic map F . Pick an affinoid neighbourhood V of $f(x)$. After maybe reducing U and taking a larger extension of k , we may assume that $f(U) \subset V$ and $g|_V = \pi_{K/k} \circ G \circ \sigma_{K/k}$ for an analytic map G . One concludes using the fact that $\sigma_{K/k}$ is a section of $\pi_{K/k}$. \square

Proposition 5.3. *Let X be a basic tube and Y be a k -affinoid space. Let $f : X \rightarrow Y$ be a continuous map. The following two conditions are equivalent.*

- i) *For any point $x \in X$ there exist an affinoid neighbourhood Z of x and a sequence of analytic maps $f_n : Z \rightarrow Y$ pointwise converging to $f|_Z$.*
- ii) *For any point $x \in X$ there exist an affinoid neighbourhood Z of x , a complete extension K of k and an analytic map $F : Z_K \rightarrow Y_K$ such that $f|_Z = \pi_{K/k} \circ F \circ \sigma_{K/k}$.*

A consequence of the previous result is that when X has no boundary then a continuous map $f : X \rightarrow Y$ is weakly analytic whenever for every point $x \in X$ there exists a basic tube U containing x and a sequence of analytic maps f_n from U to Y that converge pointwise to f .

Proof. The implication i) \Rightarrow ii) is precisely Theorem B.

Suppose that ii) is satisfied. As usual, we may assume $Y = \bar{\mathbb{D}}_k^s$. Pick a point $x \in X$ and an affinoid neighbourhood Z of x such that there exists a complete extension K/k and a K -analytic map $F : Z_K \rightarrow \bar{\mathbb{D}}_K^s$ such that $f|_Z = \pi_{K/k} \circ F \circ \sigma_{K/k}$. As in the proof of Theorem 3.7, we may find an analytic map $\hat{F} : \mathbb{D}_K^r \rightarrow \bar{\mathbb{D}}_K^s$ that agrees with F on $Z_K \cap \mathbb{D}_K^r$. By Theorem 4.6, there exists a rigid point $a \in \text{Mor}_{\infty, K}^{r, s}$ such that $\hat{F} = \Phi(a, \cdot)$. The point $\alpha = \pi_{K/k}^\infty(a)$ in $\text{Mor}_{\infty}^{r, s}$ is not rigid in general, but we may find points $\alpha^{(n)} \in \text{Mor}_{\infty}^{r, s}(k)$ converging to α . The analytic maps $\text{Ev}(\alpha^{(n)})$ converge pointwise to $\text{Ev}(\alpha) : \mathbb{D}_k^r \rightarrow \bar{\mathbb{D}}_k^s$ by Theorem 4.6, and by construction we have that $\text{Ev}(\alpha) = \pi_{K/k} \circ \hat{F} \circ \sigma_{K/k}$, see Theorem 4.9. We conclude the proof by applying the previous case. \square

5.2. Rigidity of weakly analytic maps. We prove here the following statement

Proposition 5.4. *Suppose $f : X \rightarrow Y$ is a weakly analytic map, where Y is a curve. If x is a rigid point that is mapped to a non-rigid point by f , then f is locally constant near x .*

Proof. Let $x \in X$ be a rigid point such that $y = f(x)$ is not rigid. Since this is a local statement, we may replace X and Y by affinoid neighbourhoods of x and y respectively. In particular, we may assume that $X = \mathbb{D}_k^N$ and $x = 0$. After maybe reducing X , there exists an extension K of k and a K -analytic map $F : X_K \rightarrow Y_K$ such that $f = \pi_{K/k} \circ F \circ \sigma_{K/k}$. Observe that $F(x)$ is a rigid point of Y_K .

Suppose first that $Y = \bar{\mathbb{D}}_k$. The fact that y is not rigid means that y has positive diameter, i.e.

$$\inf_{a \in k^\circ} |(T - a)(y)| = r > 0.$$

By continuity, we can find a polyradius $\epsilon > 0$ such that every rigid point z in $\mathbb{D}_K^N(0; \epsilon)$ satisfies $|F(z) - F(0)|_K < r$, where $|\cdot|_K$ denotes the absolute value on K . Pick a point $a \in k^\circ$. For every rigid point $z \in \mathbb{D}_K^N(0; \epsilon)$, we get

$$\begin{aligned} |(T - a)(y)| &= \max \{|F(z) - F(0)|_K, |(T - a)(y)|\} \\ &= \max \{|F(z) - F(0)|_K, |(T - a)(\pi_{K/k} \circ F(0))|\} \\ &= \max \{|F(z) - F(0)|_K, |F(0) - a|_K\} \\ &= |F(z) - a|_K \\ &= |(T - a)(\pi_{K/k} \circ F(z))|. \end{aligned}$$

Thus, F maps the polydisk $\mathbb{D}_K^N(0; \epsilon)$ into the fibre $\pi_{K/k}^{-1}(y)$. As

$$\sigma_{K/k}(\mathbb{D}_k^N(0; \epsilon)) \subseteq \mathbb{D}_K^N(0; \epsilon),$$

we conclude that f is locally constant near 0.

For Y any affinoid of dimension 1 there exists a finite morphism $\varphi : Y \rightarrow \bar{\mathbb{D}}_k$ by Noether's Lemma. By what precedes, the composite map $\varphi \circ f$ is locally constant near 0, and by finiteness so is f . \square

Example 5.5. *The previous result does not hold if Y has dimension greater than 2. Consider for instance the weakly analytic map $f : \bar{\mathbb{D}} \rightarrow \bar{\mathbb{D}}^2$ given by $f = \pi_{K/k} \circ F \circ \sigma_{K/k}$, where $K = \mathcal{H}(x_g)$ and $F(z) = (x_g, z)$. No rigid point in $\bar{\mathbb{D}}$ has rigid image under f , but f is not locally constant at these points.*

5.3. Weakly analytic maps from curves.

Proposition 5.6. *Let $f : X \rightarrow Y$ be a weakly analytic map, where X is a curve. If there exists a converging sequence of rigid points of X whose images under f are rigid points, then f is analytic.*

Proof. Pick any sequence $x_n \in X(k)$ such that $f(x_n)$ are also rigid, and assume that $\lim_n x_n = x$. Here x may be non-rigid. We may replace X by some affinoid neighbourhood of x and assume that $f = \pi_{K/k} \circ F \circ \sigma_{K/k}$ for some complete extension K/k and some K -analytic map F . Observe

that $f(x_n) = F(x_n) \in Y(k)$. We may as well replace Y by an affinoid neighbourhood of $f(x)$ and embed it in some polydisk $\bar{\mathbb{D}}_k^N$.

Let \mathcal{A} be the underlying k -affinoid algebra of X . The map F is then determined by elements F_1, \dots, F_N in the K -affinoid algebra \mathcal{A}_K . We may assume that the extension K/k is of countable type [BGR84, §2.7], since the expression of F contains at most countably many elements of K . Pick any real number $\alpha > 1$. By [BGR84, Proposition 2.7.2/3] there is an α -cartesian Schauder basis $\{e_j\}_{j \in \mathbb{N}}$ of K , and we may choose $e_0 = 1$ by [BGR84, Proposition 2.6.2./3].

Fix an epimorphism $\mathcal{T}_M \rightarrow \mathcal{A}_K$ and lift every F_l to an element G_l in \mathcal{T}_M . Then for every $l = 1, \dots, N$ we can develop $G_l = \sum_I a_I^l T^I$ with $a_I^l \in K$ and such that $|a_I^l|_K \rightarrow 0$ as $|I|$ goes to infinity. Using the Schauder basis we obtain $G_l = \sum_j (\sum_I a_{I,j}^l T^I) e_j$, with $a_{I,j}^l \in k$. The series $A_l^j = \sum_I a_{I,j}^l T^I$ defines an element in \mathcal{T}_M , since

$$|a_{I,j}^l|_k \leq \max_j |a_{I,j}^l|_k \leq \alpha |a_I^l|_K \rightarrow 0$$

as $|I|$ goes to infinity. Recall that $F_l(x_n) \in k$ for all n , and so $G_l(x_n) \in k$. We infer that for $j \geq 1$ and for all n , $A_l^j(x_n) = 0$. Each of these A_l^j defines in turn an analytic map on X that vanishes at every x_n , and hence is constant equal zero on X by the principle of isolated zeros. It follows that $F_{l|X} = A_l^0$ for every $1 \leq l \leq N$, thus they are defined over k . \square

We observe that the previous result does not hold in higher dimension.

Example 5.7. Let $\zeta_n \in k$, $|\zeta_n| = 1$, $|\zeta_n - \zeta_m| = 1$ for $n \neq m$. Let f be the weakly analytic map obtained as the limit of the sequence $f_n : \mathbb{D}^2 \rightarrow \bar{\mathbb{D}}^1$, given by $T \mapsto \zeta_n S + T$. The map f is not analytic, but the set $\{0\} \times \mathbb{D}^1(k)$ that is mapped to the set of rigid points.

A consequence of the previous result is the following statement that can be viewed as the principle of isolated zeroes for weakly analytic maps.

Proposition 5.8. Let $f : X \rightarrow Y$ be a non constant weakly analytic map where X is a curve without boundary. Then the fibre of any rigid point in Y contains no accumulation point.

Proof. Let $y \in Y(k)$ and suppose there exist points $x_n \in X$ converging to a point x and such that $f(x_n) = y$ for all n . In this situation, we may assume $Y = \bar{\mathbb{D}}_k^N$, $y = (0, \dots, 0)$ and replace X with some affinoid neighbourhood of x such that f lifts to a K -analytic map F over some complete extension K/k . This map F is given by some elements F_1, \dots, F_N in the underlying affinoid algebra of X_K .

The point y is rigid and so it has only one preimage under $\pi_{K/k}$. Thus,

$$(0, \dots, 0) = f(x_n) = F \circ \sigma_{K/k}(x_n) \in \bar{\mathbb{D}}_K^N$$

for all n . Since X is a curve and F is non-constant (otherwise f would be so), $F^{-1}(0)$ is included in the set of rigid points of X . It follows that every

$\sigma_{K/k}(x_n)$ is rigid. Each component F_i of F defines an analytic map between the curves X_K and $\bar{\mathbb{D}}_K$ and admits a sequence of zeros with an accumulation point $\sigma_{K/k}(x)$. It follows that every F_i is identically zero, hence so is f . \square

5.4. A conjecture on weakly analytic maps. On basic tubes, we conjecture that weakly analytic maps can be globally lifted to analytic maps.

Conjecture 1. *Let Y be a k -affinoid space and X a basic tube. Let $f : X \rightarrow Y$ be a weakly analytic map. Then, there exist a complete extension K/k and $F : X_K \rightarrow Y_K$ analytic such $f = \pi_{K/k} \circ F \circ \sigma_{K/k}$.*

Notice that a weakly analytic map can be locally lifted to an analytic map over some complete extension of k . Conjecture 1 means that this can be done globally.

Remark 5.9. *In the case when X and Y are polydisks, Conjecture 1 amounts to saying that the map Ev is surjective onto the set $\text{WA}(X, Y)$.*

The map Ev becomes closed by Theorem 4.6 for the topology of the pointwise convergence, and so $\text{WA}(X, Y)$ becomes Fréchet-Urysohn for this topology.

Observe that if Conjecture 1 holds, then using Theorem 4.7 we have:

Theorem 5.10. *Suppose that Conjecture 1 holds.*

Let X be a boundaryless σ -compact k -analytic space and Y a k -affinoid space. Then, every sequence of weakly analytic maps $f_n : X \rightarrow Y$ admits a subsequence that is pointwise converging to a weakly analytic map f .

As a consequence, we have:

Corollary 5.11. *Suppose that Conjecture 1 holds. Let X be a boundaryless σ -compact k -analytic space and Y a k -affinoid space. Let $\{f_n\} \subset \text{WA}(X, Y)$ be a sequence converging to some continuous map f . Then, f is weakly analytic.*

6. APPLICATIONS TO DYNAMICS

In this section, we define the Fatou set of an endomorphism of $\mathbb{P}^{N, \text{an}}$ and study its geometry, which exhibits similar properties to the complex case.

6.1. Strongly pluriharmonic functions. We recall the definition from [CL11]:

Definition 6.1. *Let X be an open subset of $\mathbb{P}^{N, \text{an}}$. A continuous function $u : X \rightarrow \mathbb{R}$ is strongly pluriharmonic if for every $x \in X$ there exist an open neighbourhood U of x , a sequence of invertible analytic functions h_n on U and real numbers b_n such that*

$$u = \lim_{n \rightarrow +\infty} b_n \cdot \log |h_n|$$

locally uniformly on U .

Harmonic functions have been widely studied in dimension 1. Baker-Rumely [BR10] and Favre-Rivera Letelier [FRL10], and Thuillier [Thu05] have defined non-Archimedean analogues of the Laplacian operator, on $\mathbb{P}^{1,\text{an}}$ and on general analytic curves respectively.

If X is an analytic curve, strongly harmonic functions are harmonic in the sense of Thuillier. It is not known yet whether the converse holds, see [CL11]. However, if X is a connected open subset of $\mathbb{P}^{1,\text{an}}$, then all definitions agree by [BR10, Corollary 7.32].

Observe that over \mathbb{C} , pluriharmonic functions are in fact locally the logarithm of the norm of an invertible function, whereas this is not true in the non-Archimedean setting. Counterexamples appear already for curves, see [CL11, §2.3].

Remark 6.2. *Strongly pluriharmonic functions form a sheaf by definition.*

Proposition 6.3. *Let X be an open subset of $\mathbb{P}^{N,\text{an}}$. The set of all strongly pluriharmonic functions on X forms a \mathbb{R} -vector space.*

6.2. Harmonic functions on open subsets of $\mathbb{P}^{1,\text{an}}$. Recall from [Ber90, §4.2] that the analytic projective line $\mathbb{P}^{1,\text{an}}$ is the one-point compactification of $\mathbb{A}^{1,\text{an}}$. The points in $\mathbb{A}^{1,\text{an}}$ can be explicitly described as follows [Ber90, §1.4.4].

Pick $a \in k$ and $r \in \mathbb{R}_+$ and denote by $\bar{B}(a; r)$ the closed ball in k centered at a and of radius r . To $\bar{B}(a; r)$ we can associate a point $\eta_{a,r} \in \mathbb{A}^{1,\text{an}}$ by setting $|P(\eta_{a,r})| := \sup_{|y-a| \leq r} |P(y)|$ for every polynomial $P \in k[T]$. Points of the form $\eta_{a,0}$ are called type I points, and these are precisely the rigid points of $\mathbb{A}^{1,\text{an}}$. Consider the point $\eta_{a,r}$ with $r > 0$. If $r \in |k^\times|$ we say that $\eta_{a,r}$ is of type II and if $r \notin |k^\times|$ of type III. A decreasing sequence of closed balls $\bar{B}(a_i; r_i)$ in k with empty intersection defines a converging sequence of points $\eta_{a_i, r_i} \in \mathbb{A}^{1,\text{an}}$. The limit point is called a type IV point. Any point in $\mathbb{A}^{1,\text{an}}$ is of one of these four types.

It is a fundamental fact that the Berkovich projective line carries a tree structure. Roughly speaking, it is obtained by patching together one-dimensional line segments in such a way that it contains no loop. We refer to [Jon15, §2] for a precise definition. Suffice it to say that for any two points $x, y \in \mathbb{P}^{1,\text{an}}$ there exists a closed subset $[x, y] \subset \mathbb{P}^{1,\text{an}}$ containing x and y that can be endowed with a partial order making it isomorphic to the real closed unit interval $[0, 1]$ or to $\{0\}$. These ordered sets are required to satisfy a suitable set of axioms. For instance, for any triple x, y, z there exists a unique point w such that $[z, x] \cap [y, x] = [w, x]$ and $[z, y] \cap [x, y] = [w, y]$. Any subset of the form $[x, y]$ is called a segment.

As a consequence, $\mathbb{P}^{1,\text{an}}$ is uniquely path-connected, meaning that given any two distinct points $x, y \in \mathbb{P}^{1,\text{an}}$ the image of every injective continuous map γ from the real unit interval $[0, 1]$ into $\mathbb{P}^{1,\text{an}}$ with $\gamma(0) = x$ and $\gamma(1) = y$ is isomorphic to the segment $[x, y]$.

A closed subset $\Gamma \subseteq \mathbb{P}^{1,\text{an}}$ is called a subtree if it is connected. An endpoint of Γ is a point $x \in \Gamma$ such that $\Gamma \setminus \{x\}$ remains connected. For every subtree Γ of $\mathbb{P}^{1,\text{an}}$ there is a canonical retraction $r_\Gamma : \mathbb{P}^{1,\text{an}} \rightarrow \Gamma$, which sends a point $x \in \mathbb{P}^{1,\text{an}}$ to the unique point in Γ such that the intersection of the segment $[x, r_\Gamma(x)]$ with Γ consists only of the point $r_\Gamma(x)$.

A strict finite subtree Γ of $\mathbb{P}^{1,\text{an}}$ is the convex finitely many type II points x_1, \dots, x_n . As a set, it is the union of all the paths $[x_i, x_j]$, $i = 1, \dots, n$.

Recall that an open (resp. closed) disk of $\mathbb{P}^{1,\text{an}}$ is by definition either an open (resp. closed) disk in $\mathbb{A}^{1,\text{an}}$ or $\mathbb{P}^{1,\text{an}}$ with a closed (resp. open) disk in $\mathbb{A}^{1,\text{an}}$ removed. Basic tubes in $\mathbb{P}^{1,\text{an}}$ are *strict simple domains* in the terminology of [BR10]. They are either $\mathbb{P}^{1,\text{an}}$ or strict open disks in $\mathbb{P}^{1,\text{an}}$ with a finite number of strict closed disks of $\mathbb{P}^{1,\text{an}}$ removed. In particular, basic tubes different from $\mathbb{P}^{1,\text{an}}$ and strict open disks can be obtained as an inverse image $r_\Gamma^{-1}(\Gamma^0)$, where Γ is a strict finite subtree of $\mathbb{P}^{1,\text{an}}$ and Γ^0 the open subset of Γ consisting of Γ with its endpoints removed.

Similarly, every affinoid subset of $\mathbb{P}^{1,\text{an}}$ is either a closed disk or a closed disk in $\mathbb{P}^{1,\text{an}}$ with a finite number of open disks of $\mathbb{P}^{1,\text{an}}$ removed. In particular, an affinoid subset of the form $\bar{\mathbb{D}}(a; r) \setminus \bigcup_{i=1}^n \mathbb{D}(a_i; r_i)$ is homeomorphic to the Laurent domain of underlying affinoid algebra

$$k\{r^{-1}(T - a), r_1 S_1, \dots, r_n S_n\} / (S_1(T - a_1) - 1, \dots, S_n(T - a_n) - 1).$$

Given a subset $W \subset \mathbb{P}^{1,\text{an}}$, denote by \overline{W} its closure and by $\partial_{\text{top}} W$ its topological boundary. If W is a basic tube strictly contained in $\mathbb{P}^{1,\text{an}}$, then $\partial_{\text{top}} W$ consists of a finite set of type II points.

Proposition 6.4. *Let U be a proper connected open subset of $\mathbb{P}^{1,\text{an}}$. Then there exist a sequence W_m of basic tubes of $\mathbb{P}^{1,\text{an}}$ exhausting U and a sequence of strictly affinoid subspaces X_m of $\mathbb{P}^{1,\text{an}}$ satisfying*

$$\overline{W}_m \subset X_m \subset W_{m+1} \subset U$$

for every $m \in \mathbb{N}^*$.

The proof makes extensive use of the tree structure of $\mathbb{P}^{1,\text{an}}$. Recall from [BR10, Appendix B] that the tangent space at a point $x \in \mathbb{P}^{1,\text{an}}$ is defined as the set T_x of paths leaving from x modulo the relation having a common initial segment. The space T_x is in bijection with the connected components of $\mathbb{P}^{1,\text{an}} \setminus \{x\}$. Given any tangent direction $\vec{v} \in T_x$, we denote by $U(\vec{v})$ the corresponding connected component of $\mathbb{P}^{1,\text{an}} \setminus \{x\}$.

Proof. By [BR10, Corollary 7.11] there exists a sequence of basic tubes W_m exhausting U and such that $\overline{W}_m \subset W_{m+1} \subset U$ for every $m \in \mathbb{N}^*$.

Fix a positive integer $m > 0$. As we have assumed that U is strictly contained in $\mathbb{P}^{1,\text{an}}$, the topological boundary of W_m is a non-empty finite set of type II points of $\mathbb{P}^{1,\text{an}}$. The convex hull Γ_m of $\partial_{\text{top}} W_m$ is thus a subgraph of $\mathbb{P}^{1,\text{an}}$ with finitely many endpoints.

If W_m is an open disk, we set X_m to be the closed disk of same centre and same radius as W_m . Otherwise, consider the following strict finite subtree Γ of $\mathbb{P}^{1,\text{an}}$. Let Γ_m^0 be the open subset of Γ_m consisting of Γ_m with its endpoints removed. Pick a point x in $\Gamma_m \setminus \Gamma_m^0$. There are at most finitely many tangent directions at x containing points of the complement in U and not contained in Γ_m . For every such tangent direction, attach a segment to Γ_m in that direction and in such a way that it is contained in W_{m+1} and such that its endpoint is a type II point. If no such tangent direction exists, lengthen that edge ending at x such that the new endpoint is again of type II and belongs to W_{m+1} . Denote by Γ the strict finite subtree obtained this way. Observe that all the boundary points of Γ_m are contained in Γ^0 .

Let $r_\Gamma : \mathbb{P}^{1,\text{an}} \rightarrow \Gamma$ be the natural retraction map. The basic tube W_m is precisely $r_\Gamma^{-1}(\Gamma_m^0)$. Setting $X_m = r_\Gamma^{-1}(\Gamma_m)$, clearly one has $\overline{W}_m \subset X_m \subset W_{m+1}$. Let x_{i_1}, \dots, x_{i_m} be the endpoints of Γ_m , where $x_{i_j} = \eta_{a_{i_j}, r_{i_j}}$ are of type II. The set X_m is homeomorphic to $\mathbb{P}^{1,\text{an}}$ minus the strict open disks $\mathbb{D}(a_{i_j}; r_{i_j})$, $j = 1, \dots, m$, and is thus strictly affinoid. \square

The following lemma will be essential for the proof of Theorem C.

Proposition 6.5. *Let U be a basic tube in $\mathbb{P}^{1,\text{an}}$. There exists a positive constant C depending only on U such that for every harmonic function $g : U \rightarrow \mathbb{R}$ there exists an analytic function $h : U \rightarrow \mathbb{A}^{1,\text{an}} \setminus \{0\}$ such that*

$$\sup_U |g - \log |h|| \leq C.$$

Proof. If U is either $\mathbb{P}^{1,\text{an}}$ or \mathbb{D} , the assertion is trivial, because every harmonic function on \mathbb{D} or on $\mathbb{P}^{1,\text{an}}$ is constant by [BR10, Proposition 7.12]. We may thus assume that U is of the form $\mathbb{D} \setminus \bigcup_{i=1}^m \mathbb{D}(a_i, r_i)$ with $r_i \in |k^\times|$, $0 < r_i < 1$ and $|a_i| < 1$ for $i = 1, \dots, m$. The topological boundary of U consists of a finite number of type II points $\{x_1, \dots, x_m\}$.

By the Poisson formula [BR10, Proposition 7.23], we may find real numbers c_0, \dots, c_m with $\sum_{i=1}^m c_i = 0$ such that for all $z \in U$

$$g(z) = c_0 + \sum_{i=1}^m c_i \cdot \log |(T - a_i)(z)|.$$

Pick non-zero integers n_1, \dots, n_m such that $|c_i - n_i| < 1$ and $b \in k$ such that $|\log |b| - c_0| < 1$. Consider the map $h : U \rightarrow \mathbb{A}^{1,\text{an}} \setminus \{0\}$,

$$h(z) = b \prod_{i=1}^m (T - a_i)^{n_i}(z).$$

Since $a_i \notin U$, the function $\log |h|$ is harmonic on U and we have

$$\sup_U |g - \log |h|| \leq |c_0 - \log |b|| + \sum_{i=1}^m |c_i - n_i| \cdot \sup_U \log |(T - a_i)(z)|.$$

The functions $\log |(T - a_i)(z)|$ are bounded on U and it follows that the right-hand side of the inequality is bounded. \square

6.3. Green functions after Kawaguchi-Silverman. Consider an endomorphism of the N -dimensional projective analytic space $f : \mathbb{P}_k^{N,\text{an}} \rightarrow \mathbb{P}_k^{N,\text{an}}$ of degree $d \geq 2$. Denote by f^n its n -th iterate. Fixing homogeneous coordinates, such a map can be written as $f = [F_0 : \cdots : F_N]$, with F_i homogeneous polynomials of degree d without non-trivial common zeros.

Denote by $\rho : \mathbb{A}^{N+1,\text{an}} \setminus \{0\} \rightarrow \mathbb{P}_k^{N,\text{an}}$ the natural projection map. An endomorphism f of $\mathbb{P}_k^{N,\text{an}}$ can be lifted to a map $F : \mathbb{A}^{N+1,\text{an}} \rightarrow \mathbb{A}^{N+1,\text{an}}$ such that $\rho \circ F = f \circ \rho$. One can take for instance $F = (F_0, \dots, F_N)$. In the sequel, we will always choose lifts of f such that all the coefficients of the F_i 's lie in k° and at least one of them has norm 1.

Given T_0, \dots, T_N affine coordinates of $\mathbb{A}^{N+1,\text{an}}$ and a point $z \in \mathbb{A}^{N+1,\text{an}}$, we define its norm as $|z| = \max_{0 \leq i \leq N} |T_i(z)|$. Analogously, we set $|F(z)| = \max_{0 \leq i \leq N} |F_i(z)|$. With these norms in hand, we may now define the Green function associated to f following Kawaguchi and Silverman [KS07, KS09], see [Sib99] for the complex case.

Proposition-Definition 6.6. *The sequence of functions*

$$G_n(z) = \frac{1}{d^n} \log |F^n(z)|$$

converges uniformly on $\mathbb{A}^{N+1,\text{an}}$.

One defines the dynamical Green function associated to f as $G_f(z) = \lim_{n \rightarrow \infty} G_n$.

Proof. The inequality $|F(z)| \leq |z|^d$ is clear, and by the homogeneous Nullstellensatz there exists some constant $C > 0$ such that for all z

$$C \cdot |z|^d \leq |F(z)| \leq |z|^d,$$

and so

$$C \cdot |F^n(z)|^d \leq |F^{n+1}(z)| \leq |F^n(z)|^d.$$

Set $C_1 = |\log C|$. Taking logarithms, one obtains

$$|G_{n+1} - G_n| \leq \frac{C_1}{d^n}.$$

By the ultrametric inequality, $|G_{n+j} - G_n| \leq \frac{C_1}{d^n}$ for all $j \geq 0$ and for all n . Letting j go to infinity, one obtains

$$|G_f - G_n| \leq \frac{C_1}{d^n}. \tag{6.1}$$

□

Theorem 6.7 ([KS07]). i) *The function G_f is continuous.*

ii) *For every $\lambda \in k^*$ and for every $z \in \mathbb{A}^{N+1,\text{an}}$,*

$$G_f(\lambda \cdot z) = G_f(z) + \log |\lambda|.$$

iii) *There exists a positive constant C such that*

$$\sup_{z \in \mathbb{A}^{N+1,\text{an}}} |G_f(z) - \log |z|| \leq C.$$

6.4. Fatou and Julia sets. Let us first discuss the one-dimensional situation. The Fatou and Julia sets have been widely studied, both in the complex and in the non-Archimedean setting.

Recall that there are several characterizations of the Fatou and Julia sets of an endomorphism f of $\mathbb{P}_{\mathbb{C}}^1$. The Fatou set F_f can be defined as the normality locus of the family of the iterates of f , and the Julia set J_f as its complement. Equivalently, one can set J_f to be the support of the canonical measure, see [Sib99]. One can also show that the Julia set is the closure of the repelling periodic points.

Some of these equivalences have a non-Archimedean counterpart. There is a well-defined notion of the canonical measure of an endomorphism f (see [FRL04, FRL06] and [BR10, §10.1]), and so one sets J_f to be its support and F_f its complement. Using a similar definition of normality as ours, it can be showed that the Fatou set agrees with the normality locus of the family of the iterates of f [FKT12, Theorem 5.4].

We now explore the higher dimensional case.

Definition 6.8. *The Fatou set of an endomorphism $f : \mathbb{P}^{N,\text{an}} \rightarrow \mathbb{P}^{N,\text{an}}$ of degree at least 2 is*

$$F_f = \{z \in \mathbb{P}^{N,\text{an}}; \{f^n\} \text{ is normal in some neighbourhood of } z\}.$$

The Julia set J_f is the complement of F_f .

Remark 6.9. *In [KS09], the authors define the Fatou set of an endomorphism of the N -th projective space \mathbb{P}_k^N as the equicontinuity locus of the family of iterates, which they prove to be the same as the locus where it is locally uniformly Lipschitz. However, the definition of the Fatou set in terms of equicontinuity presents some difficulties already in dimension one [BR10, Example 10.53].*

The proof of the next theorem follows its complex counterpart.

Theorem 6.10. *Let $f : \mathbb{P}^{N,\text{an}} \rightarrow \mathbb{P}^{N,\text{an}}$ be an endomorphism of degree $d \geq 2$.*

An open subset U of $\mathbb{P}^{N,\text{an}}$ lies in the Fatou set of f if and only if its Green function G_f is strongly pluriharmonic on $\rho^{-1}(U) \subset \mathbb{A}^{N+1,\text{an}}$.

Remark 6.11. *Chambert-Loir has constructed a natural invariant probability measure μ_f on $\mathbb{P}^{N,\text{an}}$ and shown that its support is contained in the locus where G_f is strongly pluriharmonic, see [CL11, Proposition 2.4.4]. Theorem 6.10 shows that the support of μ_f is included in the Julia set of f .*

Proof of Theorem 6.10. Consider a lift $F = (F_0, \dots, F_N)$ of f , where $F_i \in k[T_0, \dots, T_N]$ are homogeneous of degree d without trivial common zeros. We may assume that $\sup_{\mathbb{D}} |F(z)| = 1$. Recall from (6.1) that there exists a positive constant C_1 such that $|G_f - G_n| \leq \frac{C_1}{d^n}$ for all $n \in \mathbb{N}$.

Let U be a basic tube on which G_f is strongly pluriharmonic. Let $h_n \in \mathcal{O}_{\mathbb{A}^{N+1}}^\times(U)$ and b_n non-zero real numbers be such that G_f is the uniform

limit of the sequence $b_n \cdot \log |h_n|$. After maybe extracting a subsequence and renumbering it, we may assume that

$$|G_f - b_n \cdot \log |h_n|| \leq \frac{C_1}{d^n} \quad \forall n \gg 0.$$

Thus,

$$\begin{aligned} \left| \frac{1}{d^n} \log |F^n| - b_n \cdot \log |h_n| \right| &= \left| \frac{1}{d^n} \log \left(\frac{|F^n|}{|h_n|^{b_n \cdot d^n}} \right) \right| \\ &\leq \max \{ |G_f - b_n \cdot \log |h_n||, |G_f - G_n| \} \\ &\leq \frac{C_1}{d^n}. \end{aligned}$$

So we see that for $n \gg 0$

$$e^{-C_1} \leq \frac{|F^n|}{|h_n|^{b_n \cdot d^n}} \leq e^{C_1}. \quad (6.2)$$

Since the functions h_n have no zeros on U , each $\widetilde{F}^n = \frac{F^n}{h_n^{b_n \cdot d^n}}$ is a lift of f^n . Theorem 3.7 together with (6.2) imply that there is a subsequence of \widetilde{F}^n that is pointwise converging to some continuous map \widetilde{F} , and thus the family $\{f^n\}$ is normal.

Let z be a point in the Fatou set F_f and let U be a neighbourhood of z contained in F_f on which the family of iterates $\{f^n\}$ is normal. Let f^{n_j} be a subsequence of the iterates of f that is pointwise converging on U to some continuous map. Let $\bar{x} = \lim_{n_j} f^{n_j}(z)$. Recall that the projective space \mathbb{P}^N can be covered by a finite number of N -dimensional polydisks. After maybe rescaling the sequence, we may assume that \bar{x} lies in the chart $\{z_0 = 1, |z_i| < 2, i = 1, \dots, N\}$, and hence that so do all the $f^{n_j}(U)$'s. We thus have

$$\begin{aligned} G_{n_j} &= \frac{1}{d^{n_j}} \log |(F_0^{n_j}, \dots, F_m^{n_j})| \\ &= \frac{1}{d^{n_j}} \log |F_0^{n_j}| + \frac{1}{d^{n_j}} \log \max_{1 \leq i \leq m} \left| \frac{F_i^{n_j}}{F_0^{n_j}} \right|. \end{aligned}$$

The second term converges uniformly to 0. On the open set $\rho^{-1}(U)$, the function G_f is thus the uniform limit of the sequence $\frac{1}{d^{n_j}} \log |F_0^{n_j}|$, hence strongly pluriharmonic. \square

6.5. Hyperbolicity of the Fatou components. Recall that $\text{Mor}(X, Y)$ denotes the set of analytic maps from X to Y .

Definition 6.12. Let Ω be a relatively compact subset of an analytic space Y and U a basic tube.

The family $\text{Mor}(U, \Omega)$ is said to be normal if for every sequence of analytic maps $\{f_n\} \subset \text{Mor}(U, \Omega)$ there exists a subsequence f_{n_j} that is pointwise converging to a continuous map $f : U \rightarrow Y$.

Remark 6.13. *In the complex setting, the previous definition corresponds to the family $\text{Mor}(U, \Omega)$ being relatively compact in $\text{Mor}(U, Y)$. The definition of normality for a non-compact target is slightly different, since it allows for a sequence to be compactly divergent [Kob98, §I.3].*

Theorem C can now be reformulated in the following terms:

Theorem 6.14. *Let Ω be a Fatou component of an endomorphism $f : \mathbb{P}^{N,\text{an}} \rightarrow \mathbb{P}^{N,\text{an}}$ of degree at least 2. Let U be a connected open subset of $\mathbb{P}^{1,\text{an}}$. Then the family $\text{Mor}(U, \Omega)$ is normal.*

Proof. The projective space $\mathbb{P}^{N,\text{an}}$ can be covered by $N+1$ charts V_0, \dots, V_N homeomorphic to \mathbb{D}^N . For every $i = 0, \dots, N$, let $s_i : \{z \in \mathbb{P}^{N,\text{an}} : z_i \neq 0\} \rightarrow \mathbb{A}^{N+1,\text{an}}$ be the analytic local section of ρ sending $z = [z_0 : \dots : z_N]$ to $(\frac{z_0}{z_i}, \dots, \frac{z_{i-1}}{z_i}, 1, \frac{z_{i+1}}{z_i}, \dots, \frac{z_N}{z_i})$. Let $g : U \rightarrow \Omega$ be an analytic map. We claim that for any compact subset $K \subset U$ the map $g|_K$ admits a lift to $\rho^{-1}(\Omega)$.

Suppose first that U is not the whole $\mathbb{P}^{1,\text{an}}$. By Proposition 6.4, there exists a sequence of basic tubes W_m exhausting U and a sequence of affinoid subspaces X_m satisfying

$$\overline{W}_m \subset X_m \subset U.$$

Pick any compact subset $K \subset U$. For m sufficiently large, K is contained in some X_m . Fix $m \in \mathbb{N}^*$. Cover X_m by $\{U_i^{(m)} = \rho^{-1}(V_i) \cap X_m\}_{i=0}^N$. On every $U_{ij}^{(m)} = \rho^{-1}(V_i) \cap \rho^{-1}(V_j) \cap X_m$, we know that $\rho \circ s_i \circ g = \rho \circ s_j \circ g$, and thus $s_i \circ g = \varphi_{ij}^{(m)} \cdot (s_j \circ g)$ for some $\varphi_{ij}^{(m)} \in \mathcal{O}^\times(U_{ij}^{(m)})$. Since X_m is an affinoid of $\mathbb{P}^{1,\text{an}}$ we have that $H^1(X_m, \mathcal{O}^\times) = 0$ by [Put80]. We may thus find $\varphi_i \in \mathcal{O}^\times(U_i^{(m)})$ and $\varphi_j \in \mathcal{O}^\times(U_j^{(m)})$ such that $\varphi_{ij}^{(m)} = \frac{\varphi_i^{(m)}}{\varphi_j^{(m)}}$. On X_m consider the following local lifts of g :

$$\widehat{g}_i^m : U_i^{(m)} \rightarrow \rho^{-1}(\Omega), \quad \widehat{g}_i^m = \frac{s_i \circ g}{\varphi_i^{(m)}}.$$

It follows that $\widehat{g}_i^m = \widehat{g}_j^m$ on $U_{ij}^{(m)}$, and hence we have a lift $\widehat{g}^m : X_m \rightarrow \rho^{-1}(\Omega)$ of g as required. By Theorem 6.10 the Green function G_f of f is strongly pluriharmonic on $\rho^{-1}(\Omega)$, and thus $G_f \circ \widehat{g}^m$ is harmonic on X_m .

Let $g_n : U \rightarrow \Omega$ be a sequence of analytic maps. For every X_m consider the lifts $\widehat{g}_n^m : X_m \rightarrow \rho^{-1}(\Omega)$ of the restriction of g_n to X_m constructed above.

Fix a real number $C > 0$ and consider the set $M = \{z \in \mathbb{A}^{N+1,\text{an}} : |G_f(z)| \leq C\}$. By Theorem 6.7, the set M is compact. By Lemma 6.5, for every n and every m there exists an analytic map $h_n^m : W_m \rightarrow \mathbb{A}^{1,\text{an}} \setminus \{0\}$ such that

$$\sup_{W_m} |G_f \circ \widehat{g}_n^m - \log |h_n^m|| \leq C.$$

We set $\widetilde{g}_n^m = \frac{\widehat{g}_n^m}{h_n^m}$. Each $\widetilde{g}_n^m : W_m \rightarrow \rho^{-1}(\Omega)$ is a lift of g_n and its image lies in the compact M . By Theorem 3.7, there exists a subsequence

of \tilde{g}_n^m converging pointwise to a continuous map. By a diagonal extraction argument, we conclude that the family $\text{Mor}(U, \Omega)$ is normal.

The case $U = \mathbb{P}^{1,\text{an}}$ follows by writing $\mathbb{P}^{1,\text{an}}$ as a finite union open disks. \square

The following two propositions are closely related. Proposition 6.16 is stronger, but Proposition 6.15 will be needed for its proof.

Proposition 6.15. *Let Y be a compact k -analytic space and Ω an open subset of Y .*

If the family $\text{Mor}(\mathbb{A}^{1,\text{an}}, \Omega)$ is normal, then every analytic map $\mathbb{A}^{1,\text{an}} \rightarrow \Omega$ is constant.

Proof. Assume that the family $\text{Mor}(\mathbb{A}^{1,\text{an}}, \Omega)$ is normal. Suppose that there exists a non-constant analytic map $g : \mathbb{A}^{1,\text{an}} \rightarrow \Omega$. Consider the sequence of analytic maps from $\mathbb{A}^{1,\text{an}}$ into Ω given by $f_n(z) = g(z^n)$. By normality there is a subsequence $\{f_{n_j}\}$ that is pointwise converging to a continuous map f .

The Gauss point x_g is fixed by all the maps $z \mapsto z^n$, and so $f(x_g) = g(x_g)$. For every integer $m > 0$ let $z_m = \eta_{0,1-\frac{1}{m}} \in \mathbb{A}^{1,\text{an}}$. Since every z_m lies in the open unit disk \mathbb{D} , we have that

$$f(z_m) = \lim_{n_j \rightarrow \infty} f_{n_j}(z_m) = \lim_{n_j \rightarrow \infty} g((z_m)^{n_j}) = g(0)$$

for all m . The continuity of f implies that the $f(z_m)$ tend to $f(x_g)$ as m goes to infinity. It follows that $g(x_g) = g(0)$ is a rigid point of Ω . As the source $\mathbb{A}^{1,\text{an}}$ is one-dimensional, g must be constant. \square

Proposition 6.16. *Let Ω be an open subset of $\mathbb{P}^{N,\text{an}}$.*

If the family of analytic functions $\text{Mor}(\mathbb{A}^{1,\text{an}} \setminus \{0\}, \Omega)$ is normal, then every analytic map $\mathbb{A}^{1,\text{an}} \setminus \{0\} \rightarrow \Omega$ is constant.

Let us recall some basic topological facts. We refer to [BR10, §9] for further details. Recall from §6.2 that given a point $x \in \mathbb{P}^{1,\text{an}}$, we denote by $U(\vec{v})$ the connected component of $\mathbb{P}^{1,\text{an}} \setminus \{x\}$ corresponding to the tangent direction $\vec{v} \in T_x$.

Let $g : U \subseteq \mathbb{P}^{1,\text{an}} \rightarrow \mathbb{P}^{1,\text{an}}$ be a non-constant analytic non-constant map. For every point $x \in U$, the map g induces a tangent map T_g between T_x and $T_{g(x)}$. Let \vec{v} be a tangent direction at x that is mapped to $\vec{v}' \in T_{g(x)}$ by T_g . Then either $g(U(\vec{v})) = U(\vec{v}')$ or $g(U(\vec{v})) = \mathbb{P}^{1,\text{an}}$. This follows from the fact that the map g is open [BR10, Corollary 9.10].

Of special interest for us is the case when x is a type II point. Assume for simplicity that both x and $g(x)$ are the Gauss point. The space T_x is isomorphic to \mathbb{P}_k^1 , and the tangent map $T_g : \mathbb{P}_k^1 \rightarrow \mathbb{P}_k^1$ and can be described as follows. In homogeneous coordinates g can be written as $g = [G_0 : G_1]$ with $G_0, G_1 \in \mathcal{O}(\mathbb{A}^{1,\text{an}})$ without common zeros by [FvdP04, Theorem 2.7.6], where all the coefficients of G_0 and G_1 are of norm less or equal than one and least one has norm one. Thus, we may consider the reduction map of g ,

which is a non-constant rational map from \mathbb{P}_k^1 to itself, and hence surjective. One can show that Tg is given by the reduction of g [BR10, Corollary 9.25].

The tangent space and the tangent map can be defined analogously for $\mathbb{A}^{1,\text{an}} \setminus \{0\}$.

Proof of Proposition 6.16. Suppose that the family $\text{Mor}(\mathbb{A}^{1,\text{an}} \setminus \{0\}, \Omega)$ is normal. We first deal with the case where Ω is contained in $\mathbb{P}^{1,\text{an}}$. Let $g : \mathbb{A}^{1,\text{an}} \setminus \{0\} \rightarrow \mathbb{P}^{1,\text{an}}$ be a non-constant analytic map. We may assume that it is of the form $g = [G_0 : G_1]$ with $G_i : \mathbb{A}^{1,\text{an}} \setminus \{0\} \rightarrow \mathbb{A}^{1,\text{an}}$ analytic without common zeros by [FvdP04, Theorem 2.7.6]. Our goal is to construct sequence of analytic maps from $\mathbb{A}^{1,\text{an}} \setminus \{0\}$ to itself such that the composition with g gives a sequence $g_n : \mathbb{A}^{1,\text{an}} \setminus \{0\} \rightarrow \Omega$ that admits no converging subsequence with continuous limit.

Suppose first that there exists a type II point in $\mathbb{P}^{1,\text{an}}$ having infinitely many preimages in the segment $T = \{\eta_{0,r} \in \mathbb{A}^{1,\text{an}} : 0 < r < \infty\}$. Composing with an automorphism of $\mathbb{P}^{1,\text{an}}$, we may assume that this point is the Gauss point. Let thus $\{\eta_{0,r_n}\}$ be a sequence of preimages of x_g .

Denote by V_n the compact set containing η_{0,r_n} consisting of $\mathbb{A}^{1,\text{an}} \setminus \{0\}$ minus the open sets $U(\vec{v}_0)$ and $U(\vec{v}_\infty)$, where \vec{v}_0 and \vec{v}_∞ are the tangent directions at η_{0,r_n} pointing at 0 and at infinity respectively. As Tg is surjective, we deduce that $g(V_n)$ avoids at most two tangent directions at x_g . After maybe extracting a subsequence, we may find a connected component B of $\mathbb{P}^{1,\text{an}} \setminus \{x_g\}$ that is contained in $g(V_n)$ for all $n \gg 0$. As a consequence, we may pick a rigid point a_0 in B and rigid points $x_n \in V_n$ such that $g(x_n) = a_0$ for every $n \in \mathbb{N}$.

Consider the sequence in $\text{Mor}(\mathbb{A}^{1,\text{an}} \setminus \{0\}, \mathbb{P}^{1,\text{an}})$ given by $g_n(z) = g(x_n! z^{n!})$. By normality, we may assume that g_n converges to a continuous map g_∞ . The Gauss point x_g is fixed by g_∞ , as $g_n(x_g) = x_g$ for all $n \in \mathbb{N}$. For every fixed $n \in \mathbb{N}$ and every $m \leq n$, the map g_n sends the set of all the m -th roots of unity R_m to a_0 , and so g_∞ maps every R_m to a_0 . For every $m \in \mathbb{N}$ pick a point $\zeta_m \in R_m$ such that $\zeta_m \rightarrow x_g$ as m tends to infinity. We have that

$$g_\infty(x_g) = \lim_{m \rightarrow \infty} g_\infty(\zeta_m) = a_0,$$

contradicting the continuity.

Suppose next that every type II point in $\mathbb{P}^{1,\text{an}}$ has at most finitely many preimages in the segment T . Pick a sequence of type II points $\{\eta_{0,r_n}\}$ with $r_n \rightarrow +\infty$ as n goes to infinity. By compactness, we may assume that the points $g(\eta_{0,r_n})$ converge to some point $y_\infty \in \mathbb{P}^{1,\text{an}}$. We claim that the points $g(\eta_{0,r})$ converge to a point y_∞ as r tends to infinity. To see this, fix a basic tube V containing y_∞ . Recall that $\partial_{\text{top}} V$ is a finite set of type II points. By assumption, we have that $g(\eta_{0,r})$ does not belong to $\partial_{\text{top}} V$ for sufficiently large r . For $n \gg 0$ we have that $g(\eta_{0,r_n})$ lies in V . Thus, $g(\eta_{0,r})$ must belong to V for $r \gg 0$.

Pick any $r \in \mathbb{R}_+$ and consider the tangent direction \vec{v} at $\eta_{0,r}$ pointing towards infinity. We may assume that $g(U(\vec{v}))$ avoids at most one rigid point in $\mathbb{P}^{1,\text{an}}$, as otherwise Picard's Big theorem [CR04] asserts that g admits an analytic extension at infinity and we conclude by Proposition 6.15. After maybe varying the r_n , we may find a rigid point $a_0 \in \mathbb{P}^{1,\text{an}}$ and rigid points x_n with $|x_n| = r_n$ such that $g(x_n) = a_0$ for all n .

Consider the sequence $g_n(z) = g(x_n!z^{n!})$ and assume that it admits a continuous limit g_∞ . Our previous argument shows that g_∞ maps every set R_m to a_0 . The points $g_n(x_g)$ converge to y_∞ by our claim, and hence g_∞ is not continuous.

Assume now that Ω is an open subset of $\mathbb{P}^{N,\text{an}}$. Let $g : \mathbb{A}^{1,\text{an}} \setminus \{0\} \rightarrow \Omega$ be a non-constant analytic map. As in the one-dimensional case, this map can be written in homogeneous coordinates as $g = [G_0 : \dots : G_N]$, with $G_i \in \mathcal{O}^\times(\mathbb{A}^{1,\text{an}} \setminus \{0\})$. As g is not constant we may assume that G_0 is non-constant and that G_1 is not a scalar multiple of G_0 . We may assume by [FvdP04, Theorem 2.7.6] that G_0 and G_1 have no common zeros. As a consequence, the map defined on the image of g by

$$\pi : [G_0(z) : \dots : G_N(z)] \mapsto [G_0(z) : G_1(z)]$$

is well-defined and analytic. By construction $\pi \circ g$ is non-constant and analytic. By the previous case we may find $x_n \in k^\times$ such that no subsequence of $\{\pi \circ g(x_n!z^{n!})\}$ has a continuous limit, and thus neither $\{g(x_n!z^{n!})\}$. \square

Proof of Corollary D. It follows from Theorem 6.14 and Proposition 6.16. \square

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